Unit 9 Review

1. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$

a) System: $\begin{cases} x = -2 \\ y = 4 \\ z = 1 \end{cases}$

b) consistent

c) one solution

d) solution: $(-2, 4, 1)$

2. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$

a) system: $\begin{cases} x = -2 \\ y = 4 \\ 0 = 1 \end{cases}$

b) inconsistent

c) no solutions

d) solution set: $\emptyset$ or $\{\}$

3. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

a) System: $\begin{cases} x = -2 \\ y + 3z = 4 \\ 0 = 0 \end{cases}$

b) consistent,

c) infinitely many solutions

d) solution set: $\{(x, y, z) \mid x = -2,\ y = -3z + 4,\ z \in \mathbb{R}\}$

4. $\left[\begin{array}{ccc|c} 2 & 4 & 5 & -2 \\ 1 & 2 & 3 & 4 \\ 3 & 3 & 7 & 1 \end{array}\right] \xrightarrow{R_2 = -3r_2 + r_3} \left[\begin{array}{ccc|c} 2 & 4 & 5 & -2 \\ -3(1)+3 & -3(2)+3 & -3(3)+7 & -3(4)+1 \\ 3 & 3 & 7 & 1 \end{array}\right]$

$\longrightarrow \left[\begin{array}{ccc|c} 2 & 4 & 5 & -2 \\ 0 & -3 & -2 & -11 \\ 3 & 3 & 7 & 1 \end{array}\right]$

↑ You don't need to write this!

5. $\begin{cases} 2x - 3y = 5 \\ x + 4y = -7 \end{cases}$ $\left[\begin{array}{cc|c} 2 & -3 & 5 \\ 1 & 4 & -7 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{cc|c} 1 & 4 & -7 \\ 2 & -3 & 5 \end{array}\right]$

$\xrightarrow{R_2 = -2r_1 + r_2} \left[\begin{array}{cc|c} 1 & 4 & -7 \\ 0 & -11 & 19 \end{array}\right] \xrightarrow{R_2 = -\frac{1}{11}r_2} \left[\begin{array}{cc|c} 1 & 4 & -7 \\ 0 & 1 & -\frac{19}{11} \end{array}\right]$

$\xrightarrow{R_1 = -4r_2 + r_1} \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{11} \\ 0 & 1 & -\frac{19}{11} \end{array}\right]$

Solution: $\left(-\frac{1}{11}, -\frac{19}{11}\right)$

6. $\begin{vmatrix} 3 & 5 \\ -2 & 7 \end{vmatrix} = (3)(7) - (-2)(5) = 21 + 10 = \boxed{31}$

7. $\begin{vmatrix} 2 & -5 \\ 4 & x \end{vmatrix} = 6$

$2x - (4)(-5) = 6$

$2x + 20 = 6$

$2x = -14$

$\boxed{x = -7}$

8. $\begin{vmatrix} -1 & 2 & 1 \\ 2 & -2 & 3 \\ 3 & -1 & 0 \end{vmatrix}$

Expansion by minors:

$-1\begin{vmatrix} -2 & 3 \\ -1 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & -2 \\ 3 & -1 \end{vmatrix}$

same sign; opposite sign; same sign

$= -1(0-(-3)) - 2(0-9) + 1(-2-(-6))$

$= -1(3) - 2(-9) + 1(4) = -3 + 18 + 4 = \boxed{19}$

Diagonals: $-6 + 3 + 0 = -3$

$\begin{vmatrix} -1 & 2 & 1 & -1 & 2 \\ 2 & -2 & 3 & 2 & -2 \\ 3 & -1 & 0 & 3 & -1 \end{vmatrix}$

$0 + 18 + -2 = 16$

$16 - (-3) = \boxed{19}$

9. $\begin{cases} 2x - 4y = -2 \\ 3x + 2y = 3 \end{cases}$

$D = \begin{vmatrix} 2 & -4 \\ 3 & 2 \end{vmatrix} = 4 - (-12) = 16$

$D_x = \begin{vmatrix} -2 & -4 \\ 3 & 2 \end{vmatrix} = -4 - (-12) = 8$

$D_y = \begin{vmatrix} 2 & -2 \\ 3 & 3 \end{vmatrix} = 6 - (-6) = 12$

$x = \frac{D_x}{D} = \frac{8}{16} = \frac{1}{2}$

$y = \frac{D_y}{D} = \frac{12}{16} = \frac{3}{4}$

$\boxed{\left(\frac{1}{2}, \frac{3}{4}\right)}$

10. $\begin{cases} 2x - y + z = 3 \\ x - y - z = 4 \\ x + 2y - 2z = 1 \end{cases}$ $\quad D = \begin{vmatrix} 2 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & -2 \end{vmatrix} = 10$ ← Given by problem

$$D_z = \begin{vmatrix} 2 & -1 & 3 \\ 1 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} \begin{matrix} 2 & -1 \\ 1 & -1 \\ 1 & 2 \end{matrix}$$

$-3 + 16 + (-1) = 12$

$0 - 12 = -12$

$-2 + (-4) + 6 = 0$

$$z = \frac{D_z}{D} = \frac{-12}{10} = \boxed{\frac{-6}{5}}$$

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 4 \\ -2 & 7 \end{bmatrix}$$

11. $A + B = \begin{bmatrix} 2+(-3) & 1+4 \\ 4+(-2) & -3+7 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 5 \\ 2 & 4 \end{bmatrix}}$

12. $3A - 2B = \begin{bmatrix} 6 & 3 \\ 12 & -9 \end{bmatrix} - \begin{bmatrix} -6 & 8 \\ -4 & 14 \end{bmatrix} = \begin{bmatrix} 6-(-6) & 3-8 \\ 12-(-4) & -9-14 \end{bmatrix} = \boxed{\begin{bmatrix} 12 & -5 \\ 16 & -23 \end{bmatrix}}$

13. $I_2 - 4A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 4 \\ 16 & -12 \end{bmatrix} = \begin{bmatrix} 1-8 & 0-4 \\ 0-16 & 1-(-12) \end{bmatrix} = \boxed{\begin{bmatrix} -7 & -4 \\ -16 & 13 \end{bmatrix}}$

14. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1(1)+2(-1)+3(2) & 1(2)+2(0)+3(4) \\ 0(1)+(-1)(-1)+4(2) & 0(2)+(-1)(0)+4(4) \end{bmatrix} = \boxed{\begin{bmatrix} 5 & 14 \\ 9 & 16 \end{bmatrix}}$

$2 \times 3 \quad 3 \times 2$

2×2

15. $\begin{bmatrix} 1 & 0 \\ -4 & 5 \\ 3 & -9 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 5 \\ 11 & 2 & -7 \\ 8 & 0 & 1 \end{bmatrix}$ can't multiply

$3 \times \boxed{2 \quad 3} \times 3$

16. $A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$ $\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right]$ $\xrightarrow{R_1 = -r_2 + r_1}$ $\left[\begin{array}{cc|cc} 1 & -5 & 1 & -1 \\ 2 & 4 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_2 = -2r_1 + r_2}$ $\left[\begin{array}{cc|cc} 1 & -5 & 1 & -1 \\ 0 & 14 & -2 & 3 \end{array}\right]$ $\xrightarrow{R_2 = \frac{1}{14} r_2}$ $\left[\begin{array}{cc|cc} 1 & -5 & 1 & -1 \\ 0 & 1 & -\frac{1}{7} & \frac{3}{14} \end{array}\right]$

$\xrightarrow{R_1 = 5r_2 + r_1}$ $\left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{7} & \frac{1}{14} \\ 0 & 1 & -\frac{1}{7} & \frac{3}{14} \end{array}\right]$ $\boxed{A^{-1} = \begin{bmatrix} 2/7 & 1/14 \\ -1/7 & 3/14 \end{bmatrix}}$

- OR - Shortcut: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{|A|} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$|A| = \begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix} = 12 - (-2) = 14$

$A^{-1} = \frac{1}{14} \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} 2/7 & 1/14 \\ -1/7 & 3/14 \end{bmatrix}}$

17. $\begin{cases} x + 2y + 3z = 2 \\ x + y + z = -3 \\ -x + y + 2z = 4 \end{cases}$ $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$

$A \quad X = B$

$X = A^{-1} \cdot B$ (A^{-1} was given in problem)

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -3 & 5 & 2 \\ 2 & -3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1(2) + (-1)(-3) + (-1)(4) \\ (-3)(2) + (5)(-3) + (2)(4) \\ 2(2) + (-3)(-3) + (-1)(4) \end{bmatrix} = \begin{bmatrix} 1 \\ -13 \\ 9 \end{bmatrix}$

$\boxed{(1, -13, 9)}$

18. $(x-2)(x+4)\left(\frac{2x-1}{(x-2)(x+4)}\right) = \left(\frac{A}{x-2} + \frac{B}{x+4}\right)(x-2)(x+4)$

$2x-1 = A(x+4) + B(x-2)$

Let $x=2$: $2(2)-1 = A(2+4) + B(2-2)$

$3 = 6A$

$A = \frac{1}{2}$

Let $x=-4$: $2(-4)-1 = A(-4+4) + B(-4-2)$

$-9 = -6B$

$B = \frac{3}{2}$

$$\frac{\frac{1}{2}}{x-2} + \frac{\frac{3}{2}}{x+4}$$

19. $\frac{x}{x^2+7x+12}$

$(x+3)(x+4)\left(\frac{x}{(x+3)(x+4)}\right) = \left(\frac{A}{x+3} + \frac{B}{x+4}\right)(x+3)(x+4)$

$x = A(x+4) + B(x+3)$

Let $x=-3$: $-3 = A(-3+4) + B(-3+3)$

$-3 = A$

Let $x=-4$: $-4 = A(-4+4) + B(-4+3)$

$-4 = -B$

$B = 4$

$$\frac{-3}{x+3} + \frac{4}{x+4}$$

20. $(x+1)(x-1)^2\left(\frac{7x^2-11x-10}{(x+1)(x-1)^2}\right)=\left(\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\right)(x+1)(x-1)^2$

$$7x^2-11x-10 = A(x-1)^2+B(x+1)(x-1)+C(x+1)$$

Let $x=-1$: $7(-1)^2-11(-1)-10 = A(-1-1)^2+B(-1+1)(-1-1)+C(-1+1)$

$$8 = 4A \longrightarrow A = 2$$

Let $x=1$: $7(1)^2-11(1)-10 = A(1-1)^2+B(1+1)(1-1)+C(1+1)$

$$-14 = 2C \longrightarrow C = -7$$

Let $x=0$: $7(0)^2-11(0)-10 = A(0-1)^2+B(0+1)(0-1)+C(0+1)$

$$-10 = A-B+C$$

$$-10 = 2-B+(-7) \longrightarrow -10 = -B-5 \longrightarrow B = 5$$

$$\boxed{\frac{2}{x+1}+\frac{5}{x-1}+\frac{-7}{(x-1)^2}}$$

21. $(x-3)(x^2+3x+9)\left(\frac{20x+21}{(x-3)(x^2+3x+9)}\right)=\left(\frac{A}{x-3}+\frac{Bx+C}{x^2+3x+9}\right)(x-3)(x^2+3x+9)$

$$20x+21 = A(x^2+3x+9)+(Bx+C)(x-3)$$

Let $x=3$: $\quad 20(3)+21 = A(3^2+3(3)+9)+(B(3)+C)(3-3)$

$$81 = 27A \longrightarrow A=3$$

Let $x=0$: $\quad 20(0)+21 = A(0^2+3(0)+9)+(B(0)+C)(0-3)$

$$21 = 9A-3C$$
$$21 = 9(3)-3C$$
$$21 = 27-3C$$
$$-6 = -3C \longrightarrow C=2$$

Let $x=1$: $\quad 20(1)+21 = A(1^2+3(1)+9)+(B(1)+C)(1-3)$

$$41 = 13A+(B+C)(-2)$$
$$41 = 13A-2B-2C$$
$$41 = 13(3)-2B-2(2)$$
$$41 = -2B+35$$
$$6 = -2B \longrightarrow B=-3$$

$$\boxed{\frac{3}{x-3}+\frac{-3x+2}{x^2+3x+9}}$$