Precalculus – 5.4 Notes **Double-Angle and Half-Angle Identities**

Using the sum identities from the last section, we can derive more formulas called the double-angle identities:

$$\sin(2x) = \sin(x+x) = \sin x\cos x + \cos x\sin x$$

$$\sin(2x) = 2\sin x\cos x$$

$$\cos(2x) = \cos(x+x) = \cos x\cos x - \sin x\sin x$$

$$\cos(2x) = \boxed{\cos^2 x - \sin^2 x}$$

$$(\cos x)^2 - (\sin x)^2$$

We can use Pythagorean Identities to derive two more identities for $\cos(2x)$.

$$\sin^2 x + \cos^2 x = 1$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\cos^2 x - (1 - \cos^2 x)$$

$$\cos^2 x - 1 + \cos^2 x$$

$$= \boxed{2\cos^2 x - 1}$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$(1 - \sin^2 x) - \sin^2 x$$

$$1 - \sin^2 x - \sin^2 x$$

$$\boxed{1 - 2\sin^2 x}$$

$$\tan(2x) = \tan(x+x) =$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\tan(2x) = \frac{\tan x + \tan x}{1 - \tan x \tan x}$$

$$= \boxed{\frac{2\tan x}{1 - \tan^2 x}}$$

**Example:** Use the double angle identities to verify that $\cos(3x) = \cos^3 x - 3\cos x \sin^2 x$ is an identity.

$\cos(3x) = \cos^3 x - 3\cos x \sin^2 x$

$\cos(2x + x)$

$\cos(2x)\cos(x) - \sin(2x)\sin(x)$

$(\cos^2(x) - \sin^2(x))(\cos x) - 2\sin x \cos x \cdot \sin x$

Distribute

$\cos^3(x) - 1\cos(x)\sin^2(x) - 2\cos(x)\sin^2(x)$

$\cos^3(x) - 3\cos(x)\sin^2(x) \overset{\checkmark}{=}$

We can use the double-angle identities to derive identities for $\sin(x/2)$, $\cos(x/2)$, and $\tan(x/2)$. We call these the half-angle identities.

To get identities for $\cos(x/2)$ and $\sin(x/2)$, we solve two of the identities for $\cos(2x)$ for $\sin x$ and $\cos x$.

$$2\cos^2 x - 1 = \cos(2x)$$

ADD 1 then ÷2

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

sq root both sides

$$\cos x = \pm\sqrt{\frac{1+\cos(2x)}{2}}$$

Let $x = \frac{u}{2}$

$$\sqrt{\frac{1+\cos \cancel{2}\left(\frac{u}{\cancel{2}}\right)}{2}}$$

$$\cos\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$1 - 2\sin^2 x = \cos(2x)$$

$$-2\sin^2 x = \cos(2x) - 1$$

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\sin x = \pm\sqrt{\frac{1-\cos(2x)}{2}}$$

Since these equations work for any angle, they also work if we replace $x$ by $(x/2)$:

$$\boxed{\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}} \qquad \boxed{\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}}$$

Instead of x we want x ÷ 2
so take each x replace x with $\frac{x}{2}$

We can then use these formulas to derive formulas for $\tan\frac{x}{2}$:

$$\tan\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{1+\cos x}} \qquad \tan\frac{x}{2}=\frac{\sin x}{1+\cos x} \qquad \tan\frac{x}{2}=\frac{1-\cos x}{\sin x}$$

Memorize $\tan\left(\frac{x}{2}\right)$

$$\tan x=\frac{\sin x}{\cos x}$$

**Examples:** Use the half-angle identities to find the exact values of the following:

$\sin(67.5°)$ $\cos(11\pi/12)$ $\tan(-\pi/8)$

QI so I use +

$\sin(67.5) = \sin\frac{(67.5)2}{2}$

$\sin\left(\frac{135°}{2}\right) = \pm\sqrt{\frac{1-\cos 135°}{2}}$

$= +\sqrt{\frac{1-\cos 135}{2}}$ $\cos 135° = \frac{-\sqrt{2}}{2}$

$= \sqrt{\frac{1 - \frac{-\sqrt{2}}{2}}{2}}$

$\sqrt{\left(\frac{2}{2}+\frac{\sqrt{2}}{2}\right) \div \frac{2}{1}}$

$\sqrt{\frac{2+\sqrt{2}}{2}\cdot\frac{1}{2}}$

$\frac{\sqrt{2+\sqrt{2}}}{\sqrt{4}}$

$\boxed{\frac{\sqrt{2+\sqrt{2}}}{2}}$

$$\cos\left(\frac{11\pi}{12}\right)$$

$$\cos \frac{\left(\frac{11\pi}{12}\right)\cdot 2}{2}$$

$$\cos \frac{\left(\frac{11\pi}{6}\right)}{2}$$

$\pm$ $\frac{11\pi}{12}$ QII

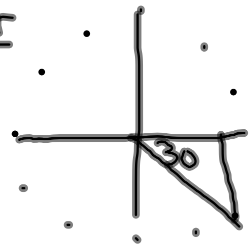
cos is neg in QII

$$x = \frac{11\pi}{6}$$

$$\cos\left(\frac{x}{2}\right) = -\sqrt{\frac{1+\cos x}{2}}$$

$$-\sqrt{\frac{1+\cos\frac{11\pi}{6}}{2}}$$

$\cos \frac{11\pi}{6}$




$$-\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

$$-\sqrt{\frac{\frac{2}{2}+\frac{\sqrt{3}}{2}}{2}}$$

$$\frac{2+\sqrt{3}}{2} \div \frac{2}{1}$$

$$-\sqrt{\frac{2+\sqrt{3}}{4}}$$

$$\frac{2+\sqrt{3}}{2} \cdot \frac{1}{2}$$

$$\boxed{-\frac{\sqrt{2+\sqrt{3}}}{2}}$$

$\tan\left(-\frac{\pi}{8}\right)$

$\tan\left(\frac{-\frac{\pi}{4}}{2}\right)$

$x = \frac{-\pi}{4}$

$\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1+\cos x}$

$$\frac{\sin\left(-\frac{\pi}{4}\right)}{1+\cos\left(-\frac{\pi}{4}\right)}$$

$$\frac{-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}$$

$$\frac{-\frac{\sqrt{2}}{2}}{\frac{2}{2}+\frac{\sqrt{2}}{2}} = \frac{-\frac{\sqrt{2}}{2}}{\frac{2+\sqrt{2}}{2}}$$

$$\frac{-\sqrt{2}}{2} \div \frac{2+\sqrt{2}}{2}$$

$$\frac{-\sqrt{2}}{\cancel{2}} \cdot \frac{\cancel{2}}{2+\sqrt{2}}$$

$$\boxed{\frac{-\sqrt{2}}{2+\sqrt{2}}}$$

**Example:** Prove that $\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)=\frac{\sin^2 x}{4}$ is an identity.

$$\left(\sqrt{\frac{1-\cos x}{2}}\right)^2\left(\sqrt{\frac{1+\cos x}{2}}\right)^2$$

$$\frac{(1-\cos x)}{2}\cdot\frac{(1+\cos x)}{2}$$

$$\frac{1-\cos^2 x}{4}$$

$$\frac{\sin^2 x}{4} \stackrel{\checkmark}{=} \frac{\sin^2 x}{4}$$

**Example:** Find $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ if $\sin(2\alpha) = -1/3$ and $\pi < 2\alpha < 3\pi/2$.

$\sin(2a) = -\frac{1}{3}$

$\frac{\pi}{2} < \frac{2a}{2} < \frac{\frac{3\pi}{2}}{2}$

$\frac{\pi}{2} < a < \frac{3\pi}{4}$

a is in QII

$\sin^2(2a) + \cos^2(2a) = 1$

$\sin^2(\theta) + \cos^2\theta = 1$

$\cos^2(2a) = 1 - \sin^2(2a)$

$\cos^2(2a) = 1 - (\sin 2a)^2$

$\cos^2(2a) = 1 - \left(-\frac{1}{3}\right)^2$

$\cos^2(2a) = 1 - \frac{1}{9}$

$\cos(2a) = \sqrt{\frac{8}{9}}$

$\cos(2a) = \frac{2\sqrt{2}}{3}$

$\cos(2a) = 2\cos^2 a - 1$

$\frac{2\sqrt{2}}{3} = 2\cos^2 a - 1$

$\left(\frac{2\sqrt{2}}{3} + 1\right) = 2\cos^2 a$

$\frac{1}{2}\left(\frac{2\sqrt{2}}{3} + \frac{3}{3}\right) = \cos^2 a$

$\boxed{\sqrt{\frac{2\sqrt{2}+3}{6}} = \cos a}$

$\cos(2a) = 1 - 2\sin^2 a$

$\frac{2\sqrt{2}}{3} = 1 - 2\sin^2 a$

$\frac{2\sqrt{2}}{3} + 1 = -2\sin^2 a$

$-\frac{1}{2}\left(\frac{2\sqrt{2}+3}{3}\right) = \sin^2 a$

$\boxed{\sqrt{\frac{-2\sqrt{2}-3}{6}} = \sin a}$

$\tan a = \frac{\sqrt{-2\sqrt{2}-3}}{\sqrt{6}} \div \frac{\sqrt{2\sqrt{2}+3}}{\sqrt{6}}$

$\frac{\sin a}{\cos a}$

$\frac{\sqrt{-2\sqrt{2}-3}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{2\sqrt{2}+3}}$

$\boxed{\frac{\sqrt{-2\sqrt{2}-3}}{\sqrt{2\sqrt{2}+3}}}$

**Example:** Find $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ if $\sin(\alpha/2) = 12/13$ and $\pi/2 < \alpha/2 < \pi$.

$$\sin\left(\frac{a}{2}\right) = \frac{12}{13}$$

$$\left(\sqrt{\frac{1-\cos a}{2}}\right)^2 = \left(\frac{12}{13}\right)^2$$

$$\frac{1-\cos a}{2} = \frac{144}{169}$$

$$\frac{\pi}{2} < \frac{a}{2} < \pi$$

$$1-\cos a = \frac{144}{169} \cdot 2$$

$$\pi < a < 2\pi$$

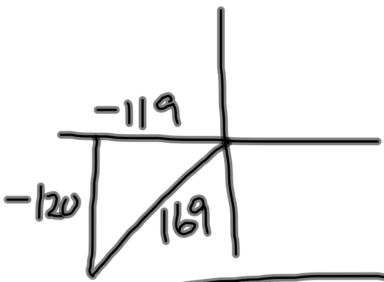
$$1-\cos a = \frac{288}{169}$$

$$-\cos a = \frac{288}{169} - 1$$

$$-\cos a = \frac{119}{169}$$

$$\boxed{\cos a = \frac{-119}{169}}$$

QIII




$$\sqrt{169^2 - (-119)^2}$$

$$\sin a = \frac{-120}{169}$$

$$\tan a = \frac{-120}{-119}$$

$$\boxed{\frac{120}{119}}$$