Precalculus - Exam 5 Review

1. $\dfrac{\tan x \csc x}{\sec x} = \dfrac{\frac{\sin x}{\cos x} \cdot \frac{1}{\sin x}}{\frac{1}{\cos x}} = \dfrac{\frac{1}{\cos x}}{\frac{1}{\cos x}} = \boxed{1}$

2. $\tan^2 x - \dfrac{\sin(-x)}{\sin x} = \tan^2 x - \dfrac{(-\sin x)}{\sin x} = \tan^2 x + 1 = \boxed{\sec^2 x}$

3. $\dfrac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} = \dfrac{\cos(2x)}{\sin(2x)} = \boxed{\cot(2x)}$

4. $\dfrac{1}{1+\sin\alpha} + \dfrac{\sin\alpha}{\cos^2\alpha} = \dfrac{1}{1+\sin\alpha} + \dfrac{\sin\alpha}{1-\sin^2\alpha}$

$= \dfrac{1}{1+\sin\alpha} + \dfrac{\sin\alpha}{(1+\sin\alpha)(1-\sin\alpha)}$

$= \left(\dfrac{1-\sin\alpha}{1-\sin\alpha}\right)\left(\dfrac{1}{1+\sin\alpha}\right) + \dfrac{\sin\alpha}{(1+\sin\alpha)(1-\sin\alpha)}$

$= \dfrac{1-\sin\alpha+\sin\alpha}{(1-\sin\alpha)(1+\sin\alpha)} = \dfrac{1}{1-\sin^2\alpha} = \dfrac{1}{\cos^2\alpha} = \boxed{\sec^2\alpha}$

5. $\dfrac{2\tan(2\theta)}{1-\tan^2(2\theta)} = \tan(2 \times 2\theta) = \boxed{\tan(4\theta)}$ ← Using $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

6. $\sin\theta\cos\theta(\tan\theta + \cot\theta)$

$= \sin\theta\cos\theta\tan\theta + \sin\theta\cos\theta\cot\theta$

$= \sin\theta\cancel{\cos\theta}\cdot\dfrac{\sin\theta}{\cancel{\cos\theta}} + \cancel{\sin\theta}\cos\theta\dfrac{\cos\theta}{\cancel{\sin\theta}}$

$= \sin^2\theta + \cos^2\theta = \boxed{1}$

7. $\sin^2 x \tan^2 x + \sin^2 x$

$= \sin^2 x(\tan^2 x + 1)$

$= \sin^2 x(\sec^2 x)$

$= \sin^2 x\left(\dfrac{1}{\cos^2 x}\right) = \dfrac{\sin^2 x}{\cos^2 x} = \boxed{\tan^2 x}$

8. $\cos 75^\circ \cos 60^\circ + \sin(-75^\circ)\sin 60^\circ$

$= \cos 75^\circ \cos 60^\circ - \sin 75^\circ \sin 60^\circ$ ← odd/even

$= \cos(75^\circ + 60^\circ)$

$= \cos(135^\circ) = \boxed{-\sqrt{2}/2}$

9. $\sin 80^\circ \cos(-50^\circ) - \sin 10^\circ \sin 50^\circ$

$= \sin 80^\circ \cos 50^\circ - \sin 10^\circ \sin 50^\circ$ ← odd/even

$= \sin 80^\circ \cos 50^\circ - \cos 80^\circ \sin 50^\circ$ ← cofunction: $\sin 10^\circ = \cos 80^\circ$

$= \sin(80^\circ - 50^\circ)$

$= \sin(30^\circ) = \boxed{\frac{1}{2}}$

10. $\dfrac{\sin(4y)}{1+\cos(4y)} = \tan\left(\dfrac{4y}{2}\right) = \boxed{\tan(2y)}$ $\left(\tan\left(\dfrac{x}{2}\right) = \dfrac{\sin x}{1+\cos x}\right)$

11. $\dfrac{\sin x \cos x}{\tan x} = 1 - \sin^2 x$

$= \dfrac{\sin x \cos x}{\frac{\sin x}{\cos x}}$

$= \cancel{\sin x} \cos x \left(\dfrac{\cos x}{\cancel{\sin x}}\right)$

$= \cos^2 x$

$= 1 - \sin^2 x$

12. $\cot(-y) = \dfrac{1-\sin^2 y}{\cos(-y)\sin(-y)}$

$= -\cot y$

$= \dfrac{1-\sin^2 y}{\cos y(-\sin y)}$

$= \dfrac{1-\sin^2 y}{-\cos y \sin y}$

$= \dfrac{\cos^2 y}{-\cos y \sin y}$

$= -\dfrac{\cos y}{\sin y}$

$= -\cot y$

13. $\dfrac{\sin(2\beta)}{2\csc\beta} = \sin^2\beta \cos\beta$

$= \dfrac{\cancel{2}\sin\beta\cos\beta}{\cancel{2}\left(\frac{1}{\sin\beta}\right)}$

$= \sin\beta\cos\beta(\sin\beta)$

$= \sin^2\beta\cos\beta$

14. $\dfrac{1}{\sec\theta - 1} - \dfrac{1}{\sec\theta + 1} = 2\cot^2\theta$

$= \left(\dfrac{\sec\theta+1}{\sec\theta+1}\right)\left(\dfrac{1}{\sec\theta-1}\right) + \left(\dfrac{-1}{\sec\theta+1}\right)\left(\dfrac{\sec\theta-1}{\sec\theta-1}\right)$

$= \dfrac{\sec\theta+1-\sec\theta+1}{\sec^2\theta-1}$

$= \dfrac{2}{\sec^2\theta-1}$

$= \dfrac{2}{\tan^2\theta}$

$= 2\cot^2\theta$

15. $\cos(3x) = \cos x(1-4\sin^2 x)$

$= \cos(2x+x)$

$= \cos(2x)\cos x - \sin(2x)\sin x$

$= (1-2\sin^2 x)\cos x - 2\sin x\cos x\sin x$

$= \cos x - 2\sin^2 x\cos x - 2\sin^2 x\cos x$

$= \cos x - 4\sin^2 x\cos x$

$= \cos x\,(1-4\sin^2 x)$

16. $\sin^2\left(\frac{x}{2}\right) = \dfrac{\csc^2 x - \cot^2 x}{2\csc^2 x + 2\csc x\cot x}$

$= \left(\pm\sqrt{\dfrac{1-\cos x}{2}}\right)^2$

$= \dfrac{1-\cos x}{2}$

$= \dfrac{1}{\frac{2}{\sin^2 x} + \frac{2}{\sin x}\cdot\frac{\cos x}{\sin x}}$

$= \dfrac{1}{\frac{2+2\cos x}{\sin^2 x}}$

$= \dfrac{\sin^2 x}{2+2\cos x}$

$= \dfrac{1-\cos^2 x}{2+2\cos x}$

$= \dfrac{\cancel{(1+\cos x)}(1-\cos x)}{2\cancel{(1+\cos x)}}$

$= \dfrac{1-\cos x}{2}$

17. $\dfrac{\sin(2\alpha)}{1+\cos(2\alpha)} = \tan\alpha$

$= \dfrac{2\sin\alpha\cos\alpha}{1+2\cos^2\alpha-1}$

$= \dfrac{2\sin\alpha\cos\alpha}{2\cos^2\alpha}$

$= \dfrac{\sin\alpha}{\cos\alpha}$

$= \tan\alpha$

18. $\sec(2\theta) = \dfrac{1+\tan^2\theta}{1-\tan^2\theta}$

$= \dfrac{1}{\cos(2\theta)}$

$= \left(\dfrac{1+\frac{\sin^2\theta}{\cos^2\theta}}{1-\frac{\sin^2\theta}{\cos^2\theta}}\right)\left(\dfrac{\cos^2\theta}{\cos^2\theta}\right)$

$= \dfrac{\cos^2\theta+\sin^2\theta}{\cos^2\theta-\sin^2\theta}$

$= \dfrac{1}{\cos(2\theta)}$

19. $\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{3\pi}{12}+\frac{4\pi}{12}\right)$
$= \cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)-\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)$
$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}$

20. $\sin\left(-\frac{5\pi}{12}\right) = \sin\left(\frac{3\pi}{12}-\frac{8\pi}{12}\right)$
$= \sin\left(\frac{\pi}{4}-\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{2\pi}{3}\right)-\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{2\pi}{3}\right)$
$= \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{-\sqrt{2}-\sqrt{6}}{4}}$

21. $\tan(195^\circ) = \tan(60^\circ+135^\circ) = \dfrac{\tan 60^\circ+\tan 135^\circ}{1-\tan 60^\circ\tan 135^\circ} = \dfrac{\sqrt{3}+(-1)}{1-(\sqrt{3})(-1)} = \boxed{\dfrac{\sqrt{3}-1}{1+\sqrt{3}}}$

22. $\sin\left(-\frac{\pi}{8}\right) = \sin\left(\frac{-\pi/4}{2}\right) = -\sqrt{\dfrac{1-\cos(-\pi/4)}{2}} = -\sqrt{\left(\dfrac{1-\sqrt{2}/2}{2}\right)\left(\dfrac{2}{2}\right)} = -\sqrt{\dfrac{2-\sqrt{2}}{4}} = \boxed{-\dfrac{\sqrt{2-\sqrt{2}}}{2}}$

QIV
sine is negative

23. $\cos(75^\circ) = \cos\left(\frac{150^\circ}{2}\right) = \sqrt{\dfrac{1+\cos(150^\circ)}{2}} = \sqrt{\left(\dfrac{1+\sqrt{3}/2}{2}\right)\left(\dfrac{2}{2}\right)} = \sqrt{\dfrac{2+\sqrt{3}}{4}} = \boxed{\dfrac{\sqrt{2+\sqrt{3}}}{2}}$

QI
cosine is positive

24. $\tan(5\pi/8) = \tan\left(\frac{5\pi/4}{2}\right) = \dfrac{1-\cos(5\pi/4)}{\sin(5\pi/4)} = \dfrac{1-\left(-\frac{\sqrt{2}}{2}\right)}{-\sqrt{2}/2} = \left(1+\frac{\sqrt{2}}{2}\right)\left(-\frac{2}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}}-1$
$= \boxed{-\sqrt{2}-1}$

25. $\sin(A+B) = \sin A\cos B + \cos A \sin B$

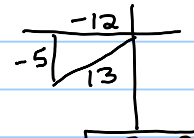
$= \left(-\frac{5}{13}\right)\left(\frac{3}{5}\right) + \left(-\frac{12}{13}\right)\left(\frac{4}{5}\right)$

$= -\frac{15}{65} - \frac{48}{65}$

$= \boxed{-\frac{63}{65}}$

$\sin A = -\frac{5}{13}$

A in QIII

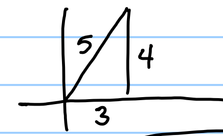


$x = -\sqrt{13^2-5^2} = -12$

$\cos A = \frac{-12}{13}$

$\cos B = \frac{3}{5}$

B in QI



$y = \sqrt{5^2-3^2} = 4$

$\sin B = \frac{4}{5}$

26. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

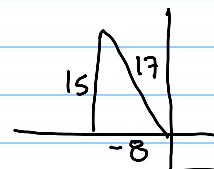
$= \left(-8/17\right)\left(2\sqrt{2}/3\right) - \left(15/17\right)\left(-1/3\right)$

$= -\frac{16\sqrt{2}}{51} + \frac{15}{51}$

$= \boxed{\frac{-16\sqrt{2}+15}{51}}$

$\sin\alpha = \frac{15}{17}$

α in QII

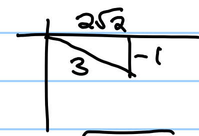


$x = -\sqrt{17^2-15^2} = -8$

$\cos\alpha = -8/17$

$\sin\beta = -\frac{1}{3}$

β in QIV



$x = \sqrt{3^2-1^2} = \sqrt{8} = 2\sqrt{2}$

$\cos\beta = 2\sqrt{2}/3$

27. $\cos\left(\frac{\alpha}{2}\right) = -\sqrt{\frac{1+\cos\alpha}{2}}$

$= -\sqrt{\left(\frac{1+\frac{\sqrt{15}}{4}}{2}\right)\left(\frac{4}{4}\right)}$

$= -\sqrt{\frac{4+\sqrt{15}}{8}}$

$= \boxed{-\frac{\sqrt{4+\sqrt{15}}}{2\sqrt{2}}}$

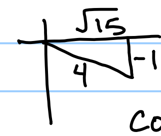
$\frac{270^\circ}{2} < \frac{\alpha}{2} < \frac{360^\circ}{2} \rightarrow 135^\circ < \alpha/2 < 180^\circ$

$\alpha/2$ in QII

$\cos(\alpha/2)$ is negative

$\sin\alpha = -\frac{1}{4}$

α in QIV



$x = \sqrt{4^2-1^2} = \sqrt{15}$

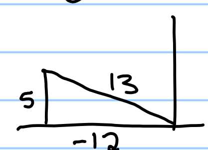
$\cos\alpha = \sqrt{15}/4$

28. $\sin(2\theta) = 2\sin\theta\cos\theta$

$\sin(2\theta) = 2\left(\frac{5}{13}\right)\left(-\frac{12}{13}\right) = \boxed{\frac{-120}{169}}$

$\cos\theta = -12/13$

θ in QII



$y = \sqrt{13^2-12^2} = 5$

$\sin\theta = \frac{5}{13}$

29. $\cos(2\beta) = 24/25$

$2\cos^2\beta - 1 = 24/25$

$2\cos^2\beta = 49/25$

$\cos^2\beta = 49/50$

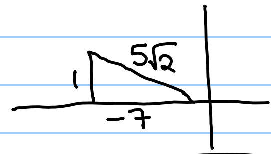
$\cos\beta = -\sqrt{49/50}$

$\boxed{\cos\beta = -\frac{7}{5\sqrt{2}}}$

$\frac{180^\circ}{2} < \frac{2\beta}{2} < \frac{360^\circ}{2} \Rightarrow 90^\circ < \beta < 180^\circ$

β in QII

$\cos\beta$ is negative



$y = \sqrt{(5\sqrt{2})^2 - 7^2} = 1$

$\boxed{\sin\beta = \frac{1}{5\sqrt{2}}}$

$\boxed{\tan\beta = -\frac{1}{7}}$

30. $\sin(\alpha/2) = -2/7$

$-\sqrt{\frac{1-\cos\alpha}{2}} = -\frac{2}{7}$

$\left(\sqrt{\frac{1-\cos\alpha}{2}}\right)^2 = \left(\frac{2}{7}\right)^2$

$\frac{1-\cos\alpha}{2} = \frac{4}{49}$

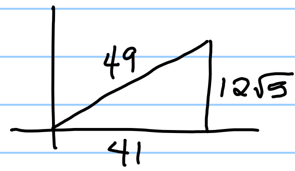
$1 - \cos\alpha = \frac{8}{49}$

$-\cos\alpha = -\frac{41}{49}$

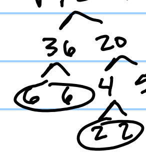
$\boxed{\cos\alpha = \frac{41}{49}}$

$2(\pi) < 2(\alpha/2) < 2(5\pi/4) \Rightarrow 2\pi < \alpha < 5\pi/2$

α in QI



$y = \sqrt{49^2 - 41^2} = \sqrt{720} = 12\sqrt{5}$



$\boxed{\sin\alpha = \frac{12\sqrt{5}}{49}}$

$\boxed{\tan\alpha = \frac{12\sqrt{5}}{41}}$