4.3 notes Calculus

Derivatives of Inverse Trigonometric Functions

Recall from our chapter 1 review that the inverse of a function can be obtained by switching x and y and solving for y. An inverse is a reflection of the function over the line y = x. Consider Figure 4.12 (The graphs of a function and its inverse; notice that the tangent lines have reciprocal slopes.)

Think of this….the slope of the function y is the reciprocal of the slope of its inverse, because you switch the x and y…..
So it is with derivatives….

The derivative idea should make sense because if we switched x and y to get y^{-1} then dy should be dx and dx → dy. $\frac{dy^{-1}}{dx} = \frac{dx}{dy}$

$\frac{dy}{dx}$

$f(a) = b$ *and* $f'(a) = c$ *and if* $g(x) = f^{-1}(x)$, *then* $g'(b) = \frac{1}{c}$

(a,b)

g(b) = a (b,a)

Example: $f(2) = 5$ *and* $f'(2) = 10$ *and if* $g(x) = f^{-1}(x)$, *then* $g'(5) = \frac{1}{10}$

(2,5)

g (5) = 2

(5,2)

$f(4) = 7$ $\qquad g(x) = f^{-1}(x)$

$f'(4) = 8$ $\qquad g(7) = 4$

$g'(7) = \frac{1}{8}$

Notice further that if the function is smooth and continuous f^{-1} will be smooth and continuous. We do have to be concerned with the domain!

Example: f(x) = sin (x)

$f^{-1}(x) = \sin^{-1} x$ Restricted domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Consider y = sin $^{-1}$ (x) what is dy/dx?
Graph the function. It has a vertical tangent, but it should be differentiable everywhere else.

$y = \sin^{-1}(x)$ angle, ratio

$\sin^2 y + \cos^2 y = 1$

* $\sin(y) = x$

$\cos^2 y = 1 - \sin^2 y$

$\cos(y) \cdot \frac{dy}{dx} = 1$

$\cos y = \sqrt{1-\sin^2 y}$

$\frac{dy}{dx} = \frac{1}{\cos y}$

$\sin y = x$

$\sin^2 y = x^2$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$

$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$ $|x| < 1$

$\boxed{\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}}$

$\boxed{\frac{d}{dx}\sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}}$

$Find\ \frac{dy}{dx}$ if $y = \sin^{-1}(x)$

$\sin y = x$ $\quad \cos y \bullet \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos y}$

We need this to be a function of x, not y.
We now do something very clever.

$$\cos^2 y = 1 - \sin^2 y \Rightarrow \cos y = \sqrt{1-\sin^2 y}$$

Sin(y) = x coordinate

So $\cos y = \sqrt{1-x^2}$

We can do this because cos y is always positive in the domain

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

If u is a differentiable function of x, we apply the chain rule and get

$$\boxed{\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad |u|<1}$$

Examples:

$y=\sin^{-1}(2x)$ *Find* $\frac{dy}{dx}$

$u = 2x$

$\frac{du}{dx} = 2$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$

$\boxed{\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}}$

$y=\sin^{-1}(1-t)$ *Find* $\frac{dy}{dt}$

$y = \sin^{-1}(1-t)$

$u = 1-t$

$\frac{du}{dt} = -1$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(1-t)^2}} \cdot -1$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(1-2t+t^2)}}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-1+2t-t^2}}$

$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{2t-t^2}}}$

MEMORIZE THESE

$$\boxed{\begin{aligned} &\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx} \\ &\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad |u|>1 \end{aligned}}$$

The derivatives of the cofunctions are negatives of one another.
Inverse Function – Inverse Cofunction Identities
Given the cofunctions find the derivatives.

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x \qquad \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x \qquad \frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$$

$$\csc^{-1}x = \frac{\pi}{2} - \sec^{-1}x \qquad \frac{d}{dx}\csc^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cos^{-1}(u) = \frac{-1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\cot^{-1}(u) = \frac{-1}{1+u^2}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\csc^{-1}(u) = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

Calculator Conversion Identities

$$\sec^{-1} x = \cos^{-1}(1/x)$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1}(x)$$

$$\csc^{-1} x = \sin^{-1}(1/x)$$

$\sec^{-1}(x) = \theta$

$\sec\theta = x$

$\cos\theta = \frac{1}{x}$

$\cos^{-1}\left(\frac{1}{x}\right) = \theta$

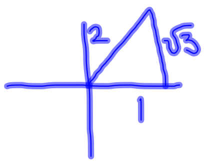
Ex $\sec^{-1}(2) = \theta$

$\sec(\theta) = 2$

$\cos(\theta) = \frac{1}{2}$

$\cos^{-1}\left(\frac{1}{2}\right) = \theta$

$\boxed{\frac{\pi}{3}}$



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$\boxed{y = \cos^{-1}(x^2)}$

cofunction with $\sin^{-1}$

$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ *so* $\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(x^2)^2}}2x = \frac{-2x}{\sqrt{1-x^4}}$ $\quad |x|<1$

$y = \cos^{-1}(x^2)$

$u = x^2 \quad \frac{du}{dx} = 2x$

$\frac{d}{dx}\cos^{-1}(u) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

$\frac{-1}{\sqrt{1-(x^2)^2}} \cdot 2x$

$\boxed{\frac{-2x}{\sqrt{1-x^4}}}$