Calculus Chapter 9 Practice Exam

1. Use L'Hôpital's rule to evaluate $\lim_{x\to 2} \frac{x^3-2x^2+x-2}{x^2+x-6}$

$$\frac{(2)^3-2(2)^2+2-2}{2^2+2-6} \quad \frac{0}{0} ✓$$

$$\lim_{x\to 2} \frac{3x^2-4x+1}{2x+1}$$

$$\frac{3(2)^2-4(2)+1}{2(2)+1} = \frac{5}{5} \Rightarrow \boxed{1}$$

2. Find $\lim_{x\to\infty} \frac{7x^2-8x+3}{4x^2+5}$ $\quad \frac{\infty}{\infty}$ ✓

$$\lim_{x\to\infty} \frac{14x-8}{8x} \quad \frac{\infty}{\infty}$$

$$\boxed{\frac{14}{8} \text{ or } \frac{7}{4}}$$

3. Use L'Hôpital's rule to find the exact value of

$$\lim_{x\to 0}\left(e^{\frac{5}{x}}-3x\right)^{\frac{x}{2}} = L$$

$$\lim_{x\to 0} \ln\left(e^{5x^{-1}}-3x\right)^{\frac{1}{2}x} = \ln L$$

back burner

$$\lim_{x\to 0} \frac{1}{2}x \ln\left(e^{5x^{-1}}-3x\right)$$

$$\lim_{x\to 0} \frac{x\ln\left(e^{5x^{-1}}-3x\right)}{2}$$

mult by x is same as dividing by $\frac{1}{x}$

$$\lim_{x\to 0} \frac{\ln\left(e^{5x^{-1}}-3x\right)}{2\cdot\frac{1}{x}}$$

$$\lim_{x\to 0} \frac{\frac{1}{e^{5x^{-1}}-3x}\cdot\left(e^{5x^{-1}}\cdot -5x^{-2}\right)-3}{\frac{-2}{x^2}}$$

$$\lim_{x\to 0} \frac{\left(-5e^{5x^{-1}}x^{-2}-3\right)}{e^{5x^{-1}}-3x}\cdot\frac{x^2}{-2}$$

stay change flip

Distribute

$$\lim_{x\to 0} \frac{-5e^{5x^{-1}}\cdot x^{0}-3x^2}{-2e^{5x^{-1}}+6x}$$

using logic

$$\lim_{x\to 0} \frac{-5e^{5x^{-1}}-3x^2}{-2e^{5x^{-1}}+6x}$$

As I use L'Hopital several more times

$\lim_{x\to 0}$ will be $\frac{5}{2}$

$$\frac{5}{2} = \ln L$$

$$\boxed{e^{\frac{5}{2}} = L}$$

4. A student attempted to use L'Hôpital's rule as follows. Identify the student's error, if any, or state "No error."

$$\lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{e^{\frac{1}{x}}}=\lim_{x\to\infty}\frac{-x^{-2}\cos\left(\frac{1}{x}\right)}{-x^{-2}e^{\frac{1}{x}}}=\lim_{x\to\infty}\frac{\cos\left(\frac{1}{x}\right)}{e^{\frac{1}{x}}}=\frac{1}{1}=1$$

$$\lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{e^{\frac{1}{x}}}\Rightarrow\frac{0}{1}$$

$\frac{0}{1}$ is not an indeterminate form

$$\lim_{x\to 0}\frac{\sin(x)}{e^{x}}\Rightarrow\frac{0}{1}$$

5. Determine which function grows faster as $x \to \infty$,

$$\ln\left(x^2+4\right) \quad or \quad x-5.$$

$$\lim_{x\to\infty} \frac{\ln(x^2+4)}{x-5} \qquad \frac{\infty}{\infty} ✓$$

$$\lim_{x\to\infty} \frac{\frac{1}{x^2+4}\cdot 2x}{1}$$

$$\lim_{x\to\infty} \frac{2x}{x^2+4} \qquad \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{2}{2x} \Rightarrow 0$$

$x-5$ grows faster than $\ln(x^2+4)$

6. Show that $f_1(x) = 5^x$, $f_2(x) = 5^{x-3}$, *and* $f_3(x) = 5^x + 3^x$ all grow at the same rate as $x \to \infty$.

Transitivity Property

$$\frac{f_2(x)}{f_1(x)} = \frac{5^{x-3}}{5^x}$$

$$\lim_{x\to\infty} \frac{\cancel{5^x} \cdot 5^{-3}}{\cancel{5^x}}$$

$$\lim_{x\to\infty} 5^{-3}$$

$$5^{-3} \text{ or } \frac{1}{5^3}$$

$f_2(x)$ and $f_1(x)$ grow at same rate

$$\frac{f_3(x)}{f_1(x)} = \frac{5^x + 3^x}{5^x}$$

$$\lim_{x\to\infty} \frac{5^x}{5^x} + \lim_{x\to\infty} \frac{3^x}{5^x}$$

$$1 + \lim_{x\to\infty} \left(\frac{3}{5}\right)^x$$

since $\frac{3}{5} < 1$

$$1 + 0$$

$$1$$

$f_3(x)$ and $f_1(x)$ grow at same rate

$$\frac{f_2(x)}{f_1(x)} \div \frac{f_3(x)}{f_1(x)}$$

$$5^{-3} \div 1$$

$\frac{f_2(x)}{f_3(x)}$ is 5^{-3}

So $f_2(x)$ and $f_3(x)$ grow at same rate

7. Order the functions e^{2x}, x^6, $3x^5$, and $(\ln x)^2$ from slowest-growing to fastest growing as $x \to \infty$.

$$\boxed{(\ln x)^2,\ 3x^5,\ x^6,\ e^{2x}}$$

try

$$\lim_{x\to\infty} \frac{(\ln x)^2}{3x^5} \quad \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{2(\ln x)^1 \frac{1}{x}}{15x^4}$$

$$\lim_{x\to\infty} \frac{\frac{2\ln x}{x}}{15x^4}$$

$$\lim_{x\to\infty} \frac{2\ln x}{15x^5} \quad \frac{\infty}{\infty}$$

$$\frac{\frac{2}{x}}{75x^4}$$

$$\lim_{x\to\infty} \frac{2}{75x^5} \Rightarrow 0$$

8. Use Partial fractions to evaluate $\int \frac{4x+30}{x^2+x-12}\,dx$

$$\frac{4x+30}{x^2+x-12} = \frac{A}{x+4} + \frac{B}{x-3}$$

$(x+4)(x-3)$

$$4x+30 = A(x-3) + B(x+4)$$

Let $x=-4$ $\quad 4(-4)+30 = A(-7)$

$$14 = -7A$$

$$-2 = A$$

Let $x=3$ $\quad 12+30 = B(7)$

$$42 = 7B$$

$$6 = B$$

$$\int \frac{-2}{x+4} + \frac{6}{x-3}\,dx$$

$$\boxed{-2\ln|x+4| + 6\ln|x-3| + C}$$

9. Use integration or the comparison test to determine whether the following integrals converge or diverge.

(a) $\int_0^\infty x^{-3}\,dx$ $\quad x \neq 0$

$$\lim_{a\to 0^+}\int_a^1 x^{-3}\,dx + \lim_{b\to\infty}\int_1^b x^{-3}\,dx$$

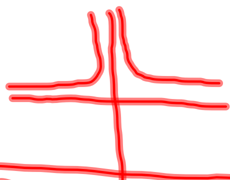
$$\lim_{a\to 0^+} \left.\frac{x^{-2}}{-2}\right|_a^1 + \lim_{b\to\infty} \left.\frac{x^{-2}}{-2}\right|_1^b$$

$$\lim_{a\to 0^+} \frac{1}{-2(1)^2} - \frac{1}{-2a^2} \qquad \lim_{b\to\infty} \cancel{\frac{1}{-2b^2}}^{\,0} - \frac{1}{-2(1)^2}$$

$$\frac{1}{2}$$

$$\lim_{a\to 0^+} \frac{1}{-2} + \frac{1}{2a^2}$$

$$\infty$$



$\boxed{\text{Diverges}}$

(b) $\int_0^{\infty} (5+\cos(x))e^{-x}\ dx$

$-1 \le \cos x \le 1$

$4 \le 5+\cos(x) \le 6$ Add 5

$4e^{-x} \le (5+\cos(x))e^{-x} \le 6e^{-x}$ mult by e^{-x}

$\int_0^{\infty} 6e^{-x}\ dx$

$\lim_{b\to\infty} \int_0^b 6e^{-x}\,dx$ $\boxed{\text{converges}}$

$\lim_{b\to\infty} -6e^{-x}\Big|_0^b$

$\lim_{b\to\infty} \cancel{\frac{-6}{e^b}}^{\,0} - \frac{-6}{e^{-0}}$

6

by DCT since $\int_0^b 6e^{-x}dx$ converges
then $\int_0^b (5+\cos(x))e^{-x}dx$ converges

(c) $\int_0^2 \frac{dx}{4-x^2}$

$\frac{1}{\underset{(2-x)(2+x)}{4-x^2}} = \frac{A}{(2-x)} + \frac{B}{(2+x)}$

$1 = A(2+x) + B(2-x)$

Let $x=2$ $\quad 1 = A(4) \quad A = \frac{1}{4}$

Let $x=-2$ $\quad 1 = B(4) \quad B = \frac{1}{4}$

$\lim_{b\to 2^-} \int_0^b \frac{\frac{1}{4}}{2-x} + \frac{\frac{1}{4}}{2+x}\ dx$

$\lim_{b\to 2^-} -\frac{1}{4}\ln|2-x| + \frac{1}{4}\ln|2+x| \Big|_0^b$

$\lim_{b\to 2^-} \frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| \Big|_0^b$

$\lim_{b\to 2^-} \frac{1}{4}\ln\left|\frac{2+b}{2-b}\right| - \frac{1}{4}\ln\left|\frac{2+0}{2-0}\right|$

∞

$\boxed{\text{diverges}}$

10. Evaluate $\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx$ or state that it diverges

$$\lim_{b\to 3^-} \int_0^b (9-x^2)^{-\frac{1}{2}} \cdot x\,dx$$

$u = 9 - x^2$

$du = -2x\,dx$

$-\frac{1}{2}\,du = x\,dx$

$u^{-\frac{1}{2}} \cdot -\frac{1}{2}\,du$ Thinking

$-\frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}}$

$-\frac{1}{2} \cdot 2u^{\frac{1}{2}}$

$-1\sqrt{u}$

$$\lim_{b\to 3^-} -\sqrt{9-x^2}\Big|_0^b$$

$$\lim_{b\to 3^-} \cancel{-\sqrt{9-b^2}}^{\,0} - -\sqrt{9-0^2}$$

$\boxed{3}$

11. Evaluate $\int_e^\infty \frac{3}{x(\ln x)^2}\,dx$ or state that it diverges

$$\lim_{b\to\infty}\int_e^b 3(\ln x)^{-2}\cdot\frac{1}{x}\,dx$$

$u=\ln x$

$du=\frac{1}{x}dx$

$$3u^{-2}\,du$$

$$\frac{3u^{-1}}{-1}$$

$$\frac{-3}{u}$$

$$\lim_{b\to\infty}\left.\frac{-3}{\ln x}\right|_e^b$$

$$\lim_{b\to\infty}\cancel{\frac{-3}{\ln b}}^{\,0}-\frac{-3}{\ln e}$$

$$\frac{3}{1}$$

$$\boxed{3}$$

12. Find the area of the region in the first quadrant that lies under the graph of $y=(3x^2+x)e^{-x}$

$$\int_0^\infty (3x^2+x)e^{-x}\,dx$$

Tabular

$\frac{d}{dx}$	$\int$
$3x^2+x$ +	e^{-x}
$6x+1$ −	$-e^{-x}$
6 +	e^{-x}
0	$-e^{-x}$

$$\lim_{b\to\infty}\int_0^b (3x^2+x)e^{-x}\,dx$$

$$-e^{-x}(3x^2+x)-e^{-x}(6x+1)+-e^{-x}(6)$$

$$-e^{-x}(3x^2+x+6x+1+6)$$

$$\lim_{b\to\infty} -e^{-x}(3x^2+7x+7)\Big|_0^b$$

$$\lim_{b\to\infty} \frac{-1}{e^x}(3x^2+7x+7)\Big|_0^b$$

$$\lim_{b\to\infty}\left[\cancelto{0}{\frac{-1}{e^b}(3b^2+7b+7)}-\frac{-1}{e^0}\left(\cancelto{0}{3(0)^2}+\cancelto{0}{7(0)}+7\right)\right]$$

$$- -1(7)$$

$$\boxed{7}$$