Calculus Chapter 9 PracticeExam

1. Use L'Hôpital's rule to evaluate $\lim_{x\to 2}\frac{x^3-2x^2+x-2}{x^2+x-6}$

$$\lim_{x\to 2}\frac{2^3-2(2)^2+2-2}{2^2+2-6} \quad \frac{0}{0}\checkmark$$

use L'Hopital

$$\lim_{x\to 2}\frac{3x^2-4x+1}{2x+1} \quad \frac{3(2)^2-4(2)+1}{2(2)+1} \quad \frac{5}{5} \quad \boxed{1}$$

2. Find $\lim_{x\to\infty}\frac{7x^2-8x+3}{4x^2+5}$ $\boxed{\frac{7}{4}}$

$$\lim_{x\to\infty}\frac{7x^2-8x+3}{4x^2+5} \quad \frac{\infty}{\infty}\checkmark$$

$$\lim_{x\to\infty}\frac{14x-8}{8x} \quad \frac{\infty}{\infty}\checkmark$$

$$\lim_{x\to\infty}\frac{14}{8} \quad \boxed{\frac{14}{8}} \text{ or } \boxed{\frac{7}{4}}$$

3. Use L'Hôpital's rule to find the exact value of

$$\lim_{x\to 0}\left(e^{\frac{5}{x}}-3x\right)^{\frac{x}{2}} = L$$

$$\lim_{x\to 0} \ln\left(e^{5x^{-1}}-3x\right)^{\frac{1}{2}x} = \ln L$$

$$\lim_{x\to 0} \frac{1}{2}x\ln\left(e^{5x^{-1}}-3x\right) = \ln L$$

$$\lim_{x\to 0} \frac{x\ln\left(e^{5x^{-1}}-3x\right)}{2}$$

MULT by x means dividing by $\frac{1}{x}$

$$\lim_{x\to 0} \frac{\ln\left(e^{5x^{-1}}-3x\right)}{2\cdot\frac{1}{x}}$$

Take the derivative (use L'Hopital)

$$\frac{\frac{1}{e^{5x^{-1}}-3x}\cdot\frac{e^{5x^{-1}}(-5x^{-2})-3}{1}}{\frac{-2}{x^2}}$$

$$\frac{\left(-5x^{-2}e^{5x^{-1}}-3\right)}{\left(e^{5x^{-1}}-3x\right)}\cdot\frac{x^2}{-2}$$

stay change flip

$$\lim_{x\to 0}\frac{-5x^0e^{5x^{-1}}-3x^2}{-2e^{5x^{-1}}+6x}$$

using L'Hopital 3 more times

$$\frac{-5}{-2}$$

$$\frac{5}{2} = \ln L$$

$$\boxed{e^{\frac{5}{2}} = L}$$

4. A student attempted to use L'Hôpital's rule as follows. Identify the student's error, if any, or state "No error."

$$\lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{e^{\frac{1}{x}}} = \lim_{x\to\infty}\frac{-x^{-2}\cos\left(\frac{1}{x}\right)}{-x^{-2}e^{\frac{1}{x}}} = \lim_{x\to\infty}\frac{\cos\left(\frac{1}{x}\right)}{e^{\frac{1}{x}}} = \frac{1}{1} = 1$$

$$\lim_{x\to 0}\frac{\sin x}{e^x} \quad \frac{\sin 0}{e^0} \quad \frac{0}{1} \qquad \boxed{0}$$

cannot use L'Hopital

$\frac{0}{1}$ is not an indeterminate form

The first step is not an indeterminate form

5. Determine which function grows faster as $x \to \infty$,

$$\ln\left(x^2+4\right) \quad or \quad x-5.$$

$$\lim_{x\to\infty} \frac{\ln(x^2+4)}{x-5} \quad \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{\frac{1}{x^2+4}\cdot 2x}{1}$$

$$\lim_{x\to\infty} \frac{2x}{x^2+4} \quad \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{2}{2x} \quad 0$$

$\boxed{x-5 \text{ grows faster}}$

6. **Show** that $f_1(x)=5^x,\quad f_2(x)=5^{x-3}, and\ f_3(x)=5^x+3^x$

all grow at the same rate as $x\to\infty$.

Transitivity

$$\frac{f_2(x)}{f_1(x)} \quad \frac{5^{x-3}}{5^x}$$

$$\lim_{x\to\infty} \frac{\cancel{5^x}\cdot 5^{-3}}{\cancel{5^x}}$$

$$\lim_{x\to\infty} 5^{-3} = \frac{1}{5^3}$$

grow at same rate

$$\frac{f_3(x)}{f_1(x)} \quad \frac{5^x+3^x}{5^x}$$

$$\lim_{x\to\infty}\frac{5^x}{5^x} + \lim_{x\to\infty}\frac{3^x}{5^x}$$

$$1 + \lim_{x\to\infty}\left(\frac{3}{5}\right)^x$$

$$1+0$$

$$1$$

grow at same rate

$$\frac{f_2(x)}{f_1(x)} \div \frac{f_3(x)}{f_1(x)}$$

$$\frac{f_2(x)}{\cancel{f_1(x)}}\cdot\frac{\cancel{f_1(x)}}{f_3(x)} = \frac{f_2(x)}{f_3(x)}$$

$$\frac{1}{5^3}\cdot 1 \qquad \frac{1}{5^3}$$

$f_2(x)$ and $f_3(x)$ grow at same rate by Transitivity property.

7. Order the functions e^{2x}, x^6, $3x^5$, and $(\ln x)^2$ from slowest-growing to fastest growing as $x \to \infty$.

$(\ln x)^2$, $3x^5$, x^6, e^{2x}

power functions exponential

log grows slower than any power function

8. Use Partial fractions to evaluate $\int \frac{4x+30}{x^2+x-12}\,dx$

$$\frac{4x+30}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$$

$$4x+30 = A(x-3)+B(x+4)$$

let $x=3$ $\quad 12+30 = 7B$

$42 = 7B \quad B = 6$

let $x=-4$ $\quad -16+30 = -7A$

$14 = -7A \quad A = -2$

$$\int \frac{-2}{x+4} + \frac{6}{x-3}\,dx$$

$$\boxed{-2\ln|x+4| + 6\ln|x-3| + C}$$

9. Use integration or the comparison test to determine whether the following integrals converge or diverge.

(a) $\int_0^\infty x^{-3}\,dx$

$$\lim_{a\to 0^+} \int_a^2 x^{-3}\,dx + \lim_{b\to\infty} \int_2^b x^{-3}\,dx$$

$$\lim_{a\to 0^+} \left.\frac{x^{-2}}{-2}\right|_a^2$$

$$\lim_{a\to 0^+} \left.\frac{1}{-2x^2}\right|_a^2$$

$$\lim_{a\to 0^+} \frac{1}{-2(2)^2} - \frac{1}{-2(0)^2}$$

$\boxed{\text{diverges}}$ no need to do the other part

(b) $\int_0^\infty (5+\cos(x))e^{-x}\ dx$

$$-1 \le \cos x \le 1$$

$$4 \le 5+\cos x \le 6 \quad \text{ADDED } 5$$

MULT by e^{-x}

$$4e^{-x} \le (5+\cos(x))e^{-x} \le 6e^{-x}$$

$$\int_0^\infty 6e^{-x}\,dx$$

$\int_0^\infty e^{-x}\,dx$ converges

by DCT since $\int_0^\infty e^{-x}dx$ converges then $\int_0^\infty (5+\cos(x))e^{-x}\,dx$ converges

(c) $\int_0^2 \frac{dx}{4-x^2}$

partial fraction decomp.

$$\frac{1}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x}$$

$$1 = A(2+x) + B(2-x)$$

let $x=2$ $\quad 1=A(2+2) \quad A=\frac{1}{4}$

let $x=-2$ $\quad 1=4B \quad B=\frac{1}{4}$

$$\lim_{b\to 2^-} \int_0^b \frac{\frac{1}{4}}{2-x} + \frac{\frac{1}{4}}{2+x}\,dx$$

$$\lim_{b\to 2^-} -\frac{1}{4}\ln|2-x| + \frac{1}{4}\ln|2+x| \Big|_0^b$$

$$\lim_{b\to 2^-}\left(-\frac{1}{4}\ln|2-b| + \frac{1}{4}\ln|2+b|\right) - \left(\frac{1}{4}\ln|2-0| + \frac{1}{4}\ln|2+0|\right)$$

diverges

10. Evaluate $\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx$ or state that it diverges

$x \neq 3$

$$\lim_{b\to 3} \int_0^b x(9-x^2)^{-\frac{1}{2}}\,dx$$

$u = 9-x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$$\int u^{-\frac{1}{2}} \cdot -\frac{1}{2}du$$

$$\frac{-\frac{1}{2}u^{\frac{1}{2}}}{\frac{1}{2}}$$

$$-\frac{1}{2}(2)\,u^{\frac{1}{2}}$$

$$-u^{\frac{1}{2}}$$

$$\lim_{b\to 3} -\sqrt{9-x^2}\,\Big|_0^b$$

$$\lim_{b\to 3} -\sqrt{\cancel{9-b^2}}^{\,\to 0} - -\sqrt{9-0^2}$$

$$\boxed{3}$$

11. Evaluate $\int_e^{\infty} \frac{3}{x(\ln x)^2}\,dx$ or state that it diverges

$$\lim_{b\to\infty} \int_e^b 3(\ln x)^{-2} \cdot \frac{1}{x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int 3u^{-2}\,du$$

$$\frac{3u^{-1}}{-1}$$

$$-3u^{-1}$$

$$\lim_{b\to\infty} \left.\frac{-3}{\ln x}\right|_e^b$$

$$\lim_{b\to\infty} \cancel{\frac{-3}{\ln b}}^{\,0} - \frac{-3}{\ln e}$$

$$\frac{3}{1}$$

$$\boxed{3}$$

12. Find the area of the region in the first quadrant that lies under the graph of $y=(3x^2+x)e^{-x}$

$$\int_0^\infty (3x^2+x)e^{-x}$$

$$\lim_{b\to\infty}\int_0^b (3x^2+x)e^{-x}$$

Tabular

$\frac{d}{dx}$	$\int$
$3x^2+x$ +	e^{-x}
$6x+1$ −	$-e^{-x}$
6 +	e^{-x}
0	$-e^{-x}$

$$-e^{-x}(3x^2+x)-e^{-x}(6x+1)+-e^{-x}(6)$$

$$\lim_{b\to\infty} -e^{-x}(3x^2+x+6x+1+6)\Big|_0^b$$

$$\lim_{b\to\infty}\left[\left(\cancel{\frac{-1}{e^b}(3(b^2)+7b+7)}\right)^0-\frac{-1}{e^0}\left(\cancel{3(0)^2}^0+\cancel{7(0)}^0+7\right)\right]$$

$$\frac{1}{1}(7)$$

$$\boxed{7}$$