Calculus BC Practice Exam Chapter 11

A calculator may be used on all problems, but answers should be written in exact form. Whenever possible, problems should be attempted analytically before using the calculator.

1. A curve is parametrized by $x = t^2 + 5$ and $y = e^{2t}$.

1. $\frac{dy}{dx} =$ $\frac{e^{2t}}{t}$

Find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ in terms of t. Express using positive exponents.

$\frac{d^2y}{dx^2} =$ ______________

$$\frac{dy}{dt} = 2e^{2t}$$

$$\frac{dx}{dt} = 2t$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \Rightarrow \frac{dy}{dx} = \frac{2e^{2t}}{2t} \Rightarrow \frac{e^{2t}}{t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\frac{\frac{d}{dt}\left(\frac{e^{2t}}{t}\right)}{\frac{dx}{dt}} = \boxed{\frac{\frac{t(2e^{2t}) - e^{2t}(1)}{t^2}}{2t}}$$

or $$\frac{2te^{2t} - e^{2t}}{t^2} \div 2t$$

$$\boxed{\frac{2te^{2t} - e^{2t}}{2t^3}}$$

2. Find the length of the curve parametrized by 2. ________

$x = \frac{1}{6}(4t+1)^{\frac{3}{2}}$, $y = t^2$, $1 \le t \le 5$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = \frac{1}{6} \cdot \frac{3}{2} (4t+1)^{\frac{3}{2}-1} \cdot 4$$

$$\frac{dx}{dt} = \frac{1}{4} \cdot 4 (4t+1)^{\frac{1}{2}}$$

$$\frac{dx}{dt} = (4t+1)^{\frac{1}{2}}$$

$$\left(\frac{dx}{dt}\right)^2 = 4t+1$$

$$\frac{dy}{dt} = 2t$$

$$\left(\frac{dy}{dt}\right)^2 = 4t^2$$

$$\int_1^5 \sqrt{4t+1+4t^2}\, dt$$

$4t^2+4t+1$ p. sq trinomial

$(2t+1)^2$

$$\int_1^5 \sqrt{(2t+1)^2}\, dt$$

$$\int_1^5 2t+1 \, dt$$

$$\frac{2t^2}{2} + 1t \Big|_1^5$$

$$(25+5) - (1^2+1)$$

$$30 - 2$$

$$\boxed{28}$$

3. Let $\boldsymbol{u} = \langle 2,-1 \rangle$ and $\boldsymbol{v} = \langle -5, 7 \rangle$

(a) Find $3\boldsymbol{u} + \boldsymbol{v}$

(b) Find the magnitude of $3\boldsymbol{u} + \boldsymbol{v}$

a)
$3\langle 2,-1\rangle + \langle -5,7\rangle$

$\langle 6, -3\rangle + \langle -5, 7\rangle$

$\langle 6+-5, -3+7\rangle$

$\boxed{\langle 1,4\rangle}$

b) $3u+v$ is $\langle 1,4\rangle$

magnitude is $\sqrt{1^2+4^2}$

$\sqrt{1+16}$

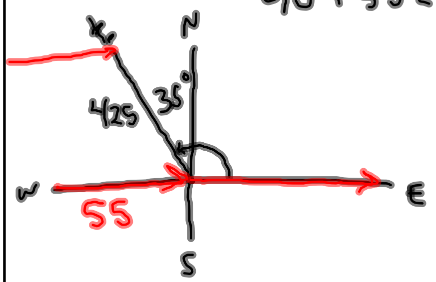
$\boxed{\sqrt{17}}$

4. An airplane, flying in the direction 35° west of north at 425 mph in still air, encounters a 55-mph wind blowing from the west (i.e. the wind direction is due east). The airplane maintains its air speed and compass heading, but, because of the wind, acquires a new ground speed and direction. What are they?

$90 + 35 = 125°$

4. Ground speed _______

Direction: ___________



$x = r\cos\theta$
$y = r\sin\theta$

plane vector $\langle 425\cos 125°, 425\sin 125° \rangle$

wind $\langle 55\cos 0°, 55\sin 0° \rangle$

plane + wind $\langle 425\cos 125 + 55, 425\sin 125° + 0 \rangle$

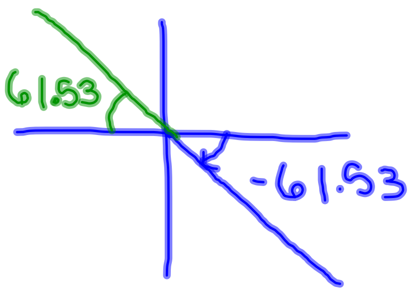
magnitude $\sqrt{(425\cos 125° + 55)^2 + (425\sin 125)^2}$

$\sqrt{(-188.769)^2 + (348.14)^2}$

magnitude = new ground speed $= \boxed{396.024 \text{ mph}}$

Direction $\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\left(\frac{425\sin 125}{425\cos 125 + 55}\right)$



$\tan^{-1}\left(\frac{348.14}{-188.769}\right) \approx -61.53°$

$90 - 61.53 = 28.47$

61.53° North of west

$\boxed{28.47° \text{ west of North}}$

5. The position vector of a particle in the plane is given by

$r(t) = \langle \ln(t+2), (t^2 - 2) \rangle$ for $-2 \leq t \leq 2$.

(a) Find the velocity vector

(b) Find the acceleration vector.

a) $v(t)$ is $\left\langle \frac{1}{t+2}, 2t \right\rangle$

b) $a(t)$ is $\left\langle \frac{-1}{(t+2)^2}, 2 \right\rangle$

$\frac{d}{dt}\left(\frac{1}{t+2}\right)$

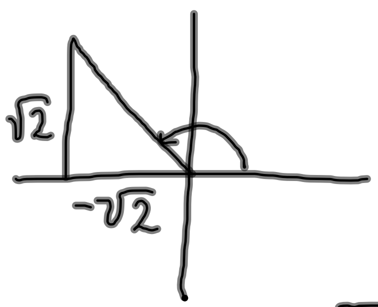
$\frac{d}{dt}\left((t+2)^{-1}\right)$

$-1(t+2)^{-2}$

$\frac{-1}{(t+2)^2}$

6. Find the magnitude of the vector and the direction angle θ it forms with the positive x-axis.

$\langle -\sqrt{2}\ , \sqrt{2} \rangle$



magnitude $\sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2}$

$\sqrt{2+2}$

$\sqrt{4}$

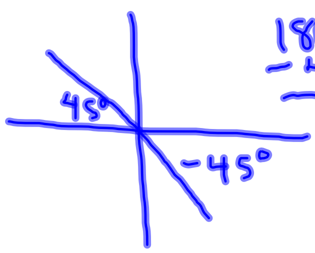
$\boxed{2}$

direction angle

$\theta = \tan^{-1}\left(\frac{\sqrt{2}}{-\sqrt{2}}\right)$

$\theta = \tan^{-1}(-1)$

$-45°$



$\begin{array}{r} 180 \\ -45 \\ \hline 135° \end{array}$

$\boxed{135°}$

or $\boxed{\frac{3\pi}{4}}$

7. Graph the polar curve given by $r = 1 + 2\cos 2\theta$

$r = 1 + 2\cos(2\theta)$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	3	-1	3	-1	3

$r(0) = 1 + 2\cos(0)$
$= 3$

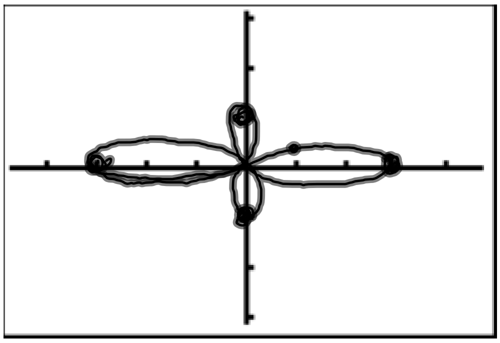
$r(\frac{\pi}{2}) = 1 + 2\cos(2\cdot\frac{\pi}{2})$
$1 + 2(-1)$
-1

$r(\pi) = 1 + 2\cos(2\pi)$
$1 + 2(1)$

$r(\frac{3\pi}{2}) = 1 + 2\cos(3\pi)$
$1 + 2(-1)$

$r(2\pi) = 3$

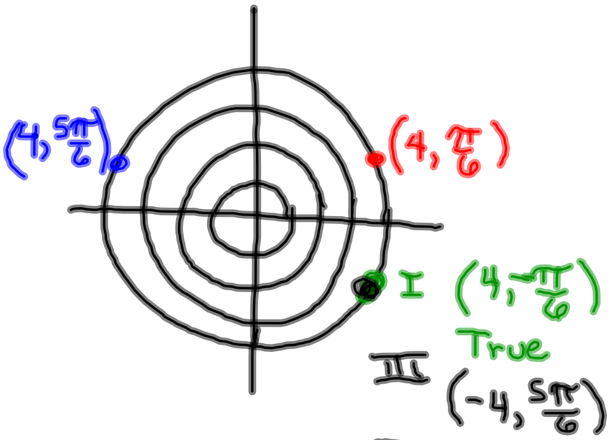
7.



8. Suppose a polar graph is symmetric about the x-axis and contains the point $\left(4, \frac{\pi}{6}\right)$. Which of the following identify another point that must be on the graph?

I. $\left(4, \frac{-\pi}{6}\right)$ II. $\left(4, \frac{5\pi}{6}\right)$ III. $\left(-4, \frac{5\pi}{6}\right)$

(A) I only (B) II only (C) III only (D) I and II (E) I and III



sym about x-axis

I and III

9. Replace the polar equation $r = \sec^2\theta$ by an equivalent Cartesian equation.

Your answer must have only x and y.
No inverse trig or trig functions

$r = \sec^2\theta$

$r = \frac{1}{\cos\theta} \cdot \frac{1}{\cos\theta}$

$\underbrace{r\cos\theta}\cdot\cos\theta = 1$

$x \cdot \cos\theta = 1$

multiply both sides by r

$x \cdot r\cos\theta = r$

$x^2 = r$

$$\boxed{x^2 = \sqrt{x^2+y^2}}$$

OR $x^4 = x^2+y^2$

we Know

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2+y^2$

$r = \sqrt{x^2+y^2}$

10. Find the slope of the polar curve $r = -2\cos 3\theta$ *at* $\theta = \frac{\pi}{6}$ *and* $\theta = \frac{\pi}{3}$.

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$r = f(\theta) = -2\cos(3\theta)$$

$$f'(\theta) = -2(-\sin(3\theta)\cdot 3)$$

$$f'(\theta) = 6\sin(3\theta)$$

$$\frac{dy}{dx} = \frac{(6\sin 3\theta)\sin\theta + (-2\cos 3\theta)\cos\theta}{(6\sin 3\theta)\cos\theta - (-2\cos 3\theta)\sin\theta}$$

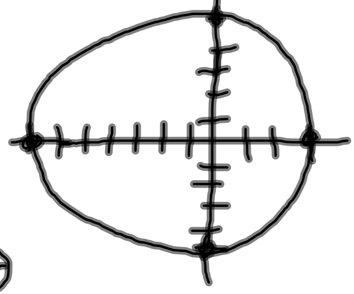
$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} \quad \frac{6\sin\left(\frac{3\pi}{6}\right)\sin\frac{\pi}{6} + \left(-2\cos\left(\frac{3\pi}{6}\right)\right)\cos\frac{\pi}{6}}{6\sin\left(\frac{3\pi}{6}\right)\cos\frac{\pi}{6} - \left(-2\cos\frac{3\pi}{6}\right)\sin\frac{\pi}{6}}$$

$$\frac{6(1)\left(\frac{1}{2}\right) - 0}{6(1)\left(\frac{\sqrt{3}}{2}\right) - 0} \quad \frac{3}{3\sqrt{3}} \quad \boxed{\frac{1}{\sqrt{3}}}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} \quad \frac{\cancel{6\sin\left(\frac{3\pi}{3}\right)}^{0}\sin\frac{\pi}{3} + \left(-2\cos\frac{3\pi}{3}\right)\cos\frac{\pi}{3}}{\cancel{6\sin\left(\frac{3\pi}{3}\right)\cos\frac{\pi}{3}}_{0} - \left(-2\cos\frac{3\pi}{3}\right)\sin\frac{\pi}{3}}$$

$$\frac{-2(-1)\frac{1}{2}}{-(-2\cdot -1)\frac{\sqrt{3}}{2}} \quad \boxed{\frac{1}{-\sqrt{3}}}$$

11. Find the area of the region enclosed by $r = 5 - 2\cos\theta$



$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$$

$$\int_0^{2\pi} \frac{1}{2}(5 - 2\cos\theta)^2 \, d\theta$$

$$\frac{1}{2}\int_0^{2\pi} 25 - 20\cos\theta + 4\cos^2\theta \, d\theta$$

$$\frac{1}{2}\int_0^{2\pi} 25 - 20\cos\theta + \overset{2}{\cancel{4}}\left(\frac{1+\cos 2\theta}{\cancel{2}}\right) d\theta$$

$$\frac{1}{2}\int_0^{2\pi} 25 - 20\cos\theta + 2 + 2\cos 2\theta \, d\theta$$

$$\frac{1}{2}\left[27\theta - 20\sin\theta + \frac{\cancel{2}\sin 2\theta}{\cancel{2}} \Big|_0^{2\pi} \right]$$

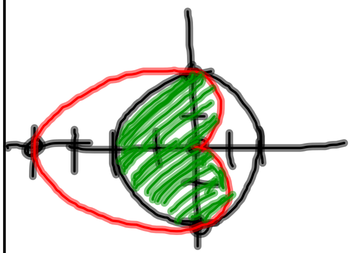
$$\frac{1}{2}\left[\left(27(2\pi) - \overset{0}{\cancel{20\sin 2\pi}} + \overset{0}{\cancel{\sin 4\pi}}\right) - \overset{0}{\left(\cancel{27(0)} - \cancel{20\sin 0} + \cancel{\sin 0}\right)}\right]$$

$$\frac{1}{2}(54\pi)$$

$$\boxed{27\pi}$$

12. Find the area of the region shared by the circle r = 2 and the cardioid r = 2(1 – cos θ).

$r = 2 - 2\cos\theta$



sym

$$\frac{1}{2}\text{circle} + 2\int_0^{\frac{\pi}{2}} \frac{1}{2}(2-2\cos\theta)^2\,d\theta$$

$$\frac{1}{2}\pi r^2 \quad + 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} 4 - 8\cos\theta + 4\cos^2\theta\,d\theta$$

$$\frac{1}{2}\pi(2)^2 + \int_0^{\frac{\pi}{2}} 4 - 8\cos\theta + \overset{2}{\cancel{4}}\left(\frac{1+\cos 2\theta}{\cancel{2}}\right)d\theta$$

$$2\pi + \int_0^{\frac{\pi}{2}} 4 - 8\cos\theta + 2 + 2\cos 2\theta\,d\theta$$

$$2\pi + \left(6\theta - 8\sin\theta + \frac{2\sin 2\theta}{2}\Big|_0^{\frac{\pi}{2}}\right)$$

$$2\pi + \left[\left(6\cdot\frac{\pi}{2}\right) - 8\sin\frac{\pi}{2} + \overset{0}{\cancel{\frac{2\sin\frac{2\pi}{2}}{2}}}\right) - \left(\overset{0}{\cancel{6(0)}} - \overset{0}{\cancel{8\sin 0}} + \overset{0}{\cancel{\frac{2\sin 0}{2}}}\right)\right]$$

$$2\pi + [3\pi - 8]$$

$$\boxed{5\pi - 8}$$