Calculus Chapter 10 Practice Exam

1. Write the first four terms of the series $\sum_{n=2}^{\infty} \frac{x^n}{3n-1}$

$n=2 \quad \frac{x^2}{3(2)-1} \quad \frac{x^2}{5}$

$n=3 \quad \frac{x^3}{3(3)-1} \quad \frac{x^3}{8}$

$n=4 \quad \frac{x^4}{3(4)-1} \quad \frac{x^4}{11}$

$n=5 \quad \frac{x^5}{3(5)-1} \quad \frac{x^5}{14}$

$$\boxed{\frac{x^2}{5} + \frac{x^3}{8} + \frac{x^4}{11} + \frac{x^5}{14}}$$

2. Tell whether the series $\sum_{n=1}^{\infty} 4\left(\frac{2}{5}\right)^n$ converges or diverges.

If it converges, find the sum.

Geometric
converge
if $|r|<1$

$r = \frac{2}{5}$

$\boxed{\text{Converges}}$

$n=1 \quad 4\left(\frac{2}{5}\right)^1 \Rightarrow \frac{8}{5}$

$n=2 \quad 4\left(\frac{2}{5}\right)^2 \Rightarrow 4 \cdot \frac{2}{5} \cdot \frac{2}{5}$

$n=3 \quad 4\left(\frac{2}{5}\right)^3 \Rightarrow 4 \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5}$

Sum is $\frac{a}{1-r}$

$$\frac{\frac{8}{5}}{1-\frac{2}{5}} \text{ OR } \boxed{\frac{4\left(\frac{2}{5}\right)}{1-\frac{2}{5}}} \text{ OR } \frac{\frac{8}{5}}{\frac{3}{5}} \Rightarrow \frac{8}{5} \cdot \frac{5}{3} \Rightarrow \boxed{\frac{8}{3}}$$

simplified

3. Given that $1 - x + x^2 + \cdots + (-x)^n + \ldots$ is a power series representation for $\frac{1}{1+x}$, find a power series representation for $\frac{x^3}{1+x^2}$.

$$\frac{1}{1+x} = 1 - x + x^2 + \ldots + (-x)^n + \ldots$$

Replace the x with x^2

$$\frac{1}{1+x^2} = 1 - (x^2) + (x^2)^2 + \ldots + (-x^2)^n + \ldots$$

or $1 - x^2 + x^4 + \ldots + (-1)^n (x^2)^n + \ldots$

Now need to muctiple by x^3

$$\frac{x^3}{1+x^2} = x^3 - (x^2)x^3 + (x^2)^2 x^3 + \ldots + (-x^2)^n x^3 + \ldots$$

or $x^3 - x^5 + x^7 + \ldots + (-1)^n (x^{2n+3}) + \ldots$

4. Find the Taylor polynomial of order 3 generated by $f(x) = sin(2x)$ at $x = \frac{\pi}{4}$

This is not centered around "0". I cannot use MacLaurin Sin(x)

center is $a = \frac{\pi}{4}$

$$f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$$

$f(x)$ $\sin(2x)\Big|_{x=\frac{\pi}{4}}$ $\sin\left(\frac{2\pi}{4}\right) = 1$

$f'(x)$ $2\cos(2x)\Big|_{x=\frac{\pi}{4}}$ $2\cos\left(\frac{2\pi}{4}\right) = 0$

$f''(x)$ $-4\sin(2x)\Big|_{x=\frac{\pi}{4}}$ $-4\sin\left(\frac{2\pi}{4}\right) = -4$

$f'''(x)$ $-8\cos(2x)\Big|_{x=\frac{\pi}{4}}$ 0

$$1 + 0\left(x-\frac{\pi}{4}\right) + \frac{-4\left(x-\frac{\pi}{4}\right)^2}{2!} + \frac{0\left(x-\frac{\pi}{4}\right)^3}{3!}$$

OR

$$1 - \frac{4\left(x-\frac{\pi}{4}\right)^2}{2!} \text{ or } 1-2\left(x-\frac{\pi}{4}\right)^2$$

5. Let f be a function that has derivatives of all orders for all real numbers. Assume $f(0) = 5$, $f'(0) = -3$, $f''(0) = 8$, $f'''(0) = 24$. Write the third order Taylor polynomial for f at x = 0 and use it to approximate $f(0.4)$

On calculator section

$$f(0) + f'(0)(x-0) + \frac{f''(0)(x-0)^2}{2!} + \frac{f'''(0)(x-0)^3}{3!}$$

$$5 + -3(x) + \frac{8(x)^2}{2!} + \frac{24(x)^3}{3!} \qquad 3! = 6$$

$$\boxed{5 - 3x + 4x^2 + 4x^3}$$

$$f(.4) \approx 5 - 3(.4) + 4(.4)^2 + 4(.4)^3$$

$$\approx \boxed{4.696}$$

6. The Maclaurin series for f(x) is

$$1 + 2x + \frac{3x^2}{2} + \frac{4x^3}{6} + \cdots + \frac{(n+1)x^n}{n!} + \cdots$$

centered $a=0$

a. Find $f''(0)$.

b. Let $g(x) = xf'(x)$. Write the Maclaurin series for g(x).

c. Let $h(x) = \int_0^x f(t)dt$. Write the Maclaurin series for h(x).

a) $\boxed{\frac{f''(0)(x-0)^2}{2!}}$

$\frac{f''(0)x^2}{2} = \frac{3x^2}{2}$

$\frac{\cancel{2}f''(0)\cancel{x^2}}{\cancel{2}\cancel{x^2}} = \frac{6x^2}{2x^2}$

$\boxed{f''(0)=3}$

OR

$f'(x)$ $\quad 2 + \frac{3\cdot\cancel{2}x}{\cancel{2}} + \frac{4\cdot 3x^2}{6} + \cdots + \frac{n(n+1)x^{n-1}}{n!} + \cdots$

$f''(x)$ $\quad 3 + 4x + \cdots + \frac{n(n+1)(n-1)x^{n-2}}{n!} + \cdots$

$f''(0) = 3$

b) $g(x) = xf'(x)$

$g(x) = x\left[2 + 3x + 2x^2 + \cdots + \frac{n(n+1)x^{n-1}}{n!} + \cdots\right]$

$g(x) = 2x + 3x^2 + 2x^3 + \cdots + \frac{n(n+1)x^{n-1}\cdot x}{n!} + \cdots$

OR $g(x) = 2x + 3x^2 + 2x^3 + \cdots + \frac{n(n+1)x^n}{n!} + \cdots$

c) $h(x) = \int_0^x f(t)\,dt$

$\int_0^x 1 + 2t + \frac{3t^2}{2} + \frac{4t^3}{6} + \cdots + \frac{(n+1)t^n}{n!} + \cdots dt$

$t + \frac{2t^2}{2} + \frac{3t^3}{2\cdot 3} + \frac{4t^4}{6\cdot 4} + \cdots + \frac{(n+1)t^{n+1}}{n!(n+1)} + \cdots \Big|_0^x$

$\boxed{x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \cdots + \frac{x^{n+1}}{n!} + \cdots}$

7. Find the Taylor polynomial of order 4 for

$f(x) = \ln(1 - x^2)$ at x = 0 and use it to approximate f(0.3)

MacLaurin

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

need

$\ln(1-x^2)$

Replace x with $-x^2$

$$\ln(1-x^2) \approx -x^2 - \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{3} - \frac{(-x^2)^4}{4}$$

This is WRONG

Because it says find the Taylor Poly of order 4 for $f(x) = \ln(1-x^2)$

I have too many terms

order 4 for $f(x) = \ln(1-x^2)$ is

$$\boxed{-x^2 - \frac{x^4}{2}}$$

$f(.3)$ $\quad -(.3)^2 - \frac{(.3)^4}{2}$

$$\boxed{-.09405}$$

8. The polynomial $1 + 7x + 21x^2$ is used to approximate $f(x) = (1 + x)^7$ on the interval $-0.01 \leq x \leq 0.01$.

Use the Remainder Estimation Theorem to estimate the maximum absolute error.

centered $a=0$

calculator section

$f(x) = (1+x)^7$ POLY $1+7x+21x^2$

$$|R_2(x)| \leq \left|\frac{f'''(c)(x-0)^3}{3!}\right|$$

$f(x) = (1+x)^7$

$f'(x) = 7(1+x)^6$

$f''(x) = 42(1+x)^5$

$f'''(x) = 210(1+x)^4$

$$|R_2(x)| \leq \frac{210(1+x)^4 x^3}{3!}$$

$$\leq \frac{210(1+.01)^4(.01)^3}{6}$$

$$\leq \boxed{.0000364211 \quad 3.642 \times 10^{-5}}$$

Let $x = (-.01)$ $f'''(-.01)$

$x = (.01)$ $f'''(.01) = 218.526842$

9. Determine the convergence or divergence of each series. Identify the test(s) you use.

a. $\sum_{n=2}^{\infty} \frac{(2n)!}{(n-1)3^n}$ $\quad \sum_{n=2}^{\infty} \frac{(2n)!}{(n-1)3^n}$

$\boxed{\text{Ratio test}}$

$$\lim_{n\to\infty} \left| \frac{(2(n+1))!}{(n+1-1)3^{n+1}} \cdot \frac{(n-1)3^n}{(2n)!} \right|$$

$(2n+2)!$ $\quad n \cdot 3^n \cdot 3^1$

$$\lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{(2n)!}\,(n-1)\cancel{3^n}}{n \cdot \cancel{3^n} \cdot 3^1 \cdot \cancel{(2n)!}}$$

$$\lim_{n\to\infty} \frac{(2n+2)(2n+1)}{3} > 1 \quad \boxed{\text{diverges}}$$

b. $\sum_{n=1}^{\infty} \frac{(n^2+3n-4)}{n!}$

Ratio test $\quad \lim_{n\to\infty} \left| \frac{(n+1)^2 + 3(n+1) - 4}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{n^2+3n-4} \right|$

degree 2 $\quad (n+1)\cancel{(n)!} \quad$ degree 2

$$\lim_{n\to\infty} \Rightarrow \frac{1}{n+1} \Rightarrow 0 \quad \boxed{\text{converges}}$$

c. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{2n}\right)^{3n}$

Ratio test $\quad \lim_{n\to\infty} \left| \frac{\left(1+\frac{1}{2(n+1)}\right)^{3(n+1)}}{\left(1+\frac{1}{2n}\right)^{3n}} \right|$

$$\lim_{n\to\infty} \frac{\left(1+\cancelto{0}{\frac{1}{2n+2}}\right)^{3n} \cdot \left(1+\cancelto{0}{\frac{1}{2n+2}}\right)^{3}}{\left(1+\cancelto{0}{\frac{1}{2n}}\right)^{3n}} = 1$$

inconclusive

$$\sum_{n=1}^{\infty} \left(\left(1+\frac{1}{2n}\right)^3\right)^n$$

Geometric

$$\left(1+\frac{1}{2n}\right)^3 > 1$$

diverges

10. Find the radius of convergence of each power series.

a. $\sum_{n=0}^{\infty} \frac{(5x)^n}{3^n}$

Geometric $\sum_{n=0}^{\infty} \left(\frac{5x}{3}\right)^n$

$-1 < \frac{5x}{3} < 1$

$-\frac{3}{5} < \frac{\cancel{5}x}{\cancel{5}} < \frac{3}{5}$

$\left(-\frac{3}{5}, \frac{3}{5}\right)$ $\boxed{R = \frac{3}{5}}$

b. $\sum_{n=1}^{\infty} \frac{n^2(2x-3)^n}{6^n}$

$\lim_{n\to\infty} \left| \frac{\cancel{(n+1)^2}\,\cancel{(2x-3)^{n+1}}}{\cancel{6^{n+1}}} \cdot \frac{\cancel{6^n}}{\cancel{n^2}\,\cancel{(2x-3)^n}} \right|$ $(2x-3)$ 6

$\lim_{n\to\infty} \frac{2x-3}{6}$ will converge if

$-1 < \frac{2x-3}{6} < 1$

$-6 < 2x-3 < 6$

$-3 < 2x < 9$

$-\frac{3}{2} < x < \frac{9}{2}$ $\boxed{R = 3}$

$\left(-1\tfrac{1}{2}, 4\tfrac{1}{2}\right)$

11. Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(4x-3)^{3n}}{8^n}$ and within this interval, the sum of the series as a function of x.

Geometric $r = \frac{(4x-3)^3}{8}$

$-1 < \frac{(4x-3)^3}{8} < 1$

$-8 < (4x-3)^3 < 8$

$-2 < 4x-3 < 2$
$+3 \quad +3 \quad +3$

$1 < 4x < 5$

$\frac{1}{4} < x < \frac{5}{4}$

Test end points

$\sum_{n=0}^{\infty} \frac{(4x-3)^{3n}}{8^n}$

$\sum_{n=0}^{\infty} \frac{(4(\frac{5}{4})-3)^{3n}}{8^n}$

$\frac{(2)^{3n}}{2^{3n}}$

$\sum_{n=0}^{\infty} 1^n$ diverges

$\sum_{n=0}^{\infty} \frac{(4(\frac{1}{4})-3)^{3n}}{8^n}$ $\frac{(-2)^{3n}}{8^n}$

$\left(\frac{-8}{8}\right)^n$

$\sum_{n=0}^{\infty} (-1)^n$ diverges

Interval of converges

$\left(\frac{1}{4}, \frac{5}{4}\right)$

Sum $\frac{a}{1-r}$ $\boxed{\frac{1}{1-\left(\frac{(4x-3)^3}{8}\right)}}$

12. Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{\sqrt{n+2} \cdot 2^n}$

$$\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{\sqrt{n+2} \cdot 2^n}$$

$$\lim_{n\to\infty} \left| \frac{\cancel{3^{n+1}}^{\,3}\,\cancel{(x-2)^{n+1}}^{\,(x-2)}}{\cancel{\sqrt{n+1+2}}\cdot \cancel{2^{n+1}}_{\,2}} \cdot \frac{\cancel{\sqrt{n+2}}\cdot\cancel{2^n}}{\cancel{3^n}\,\cancel{(x-2)^n}} \right|$$

$$-1 < \frac{3(x-2)}{2} < 1$$

$$-2 < 3(x-2) < 2$$

$$-2 < 3x-6 < 2$$

$$4 < 3x < 8$$

$$\frac{4}{3} < x < \frac{8}{3}$$

$$\boxed{\left[\frac{4}{3}, \frac{8}{3}\right)}$$

Test end points

$$\sum_{n=1}^{\infty} \frac{3^n\left(\frac{4}{3}-2\right)^n}{\sqrt{n+2}\cdot 2^n}$$

$$\frac{3^n \cdot \left(-\frac{2}{3}\right)^n}{\sqrt{n+2}\cdot 2^n} = \frac{\cancel{(-2)^n}}{\sqrt{n+2}\cdot\cancel{2^n}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$$

alt series test converges

$$\sum_{n=1}^{\infty} \frac{3^n\left(\frac{8}{3}-2\right)^n}{\sqrt{n+2}\cdot 2^n}$$

$$\frac{3^n\left(\frac{2}{3}\right)^n}{\sqrt{n+2}\cdot 2^n} \Rightarrow \cancel{\frac{2^n}{\sqrt{n+2}\cdot 2^n}}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$$

p series integral diverges