9.4 notes calculus

# Improper Integrals

The name is awful, but as it turns out most of the time there is nothing wrong with the integrals we call improper. The following is an improper integral.

**Example**: $\int_0^\infty e^{-x}\,dx$ Improper because of ∞

$$\xcancel{-e^{-x}\Big|_0^{\infty}}$$

$$\xcancel{-e^{-\infty} - -e^{-0}}$$

Do not plug in infinity.

Why ∞ is not an upper limit

not a number

$$\int_0^4 e^{-x}\,dx$$

$$-e^{-x}\Big|_0^4$$

$$-e^{-4} - -e^{-0}$$

$$\frac{-1}{e^4} + 1$$

The problem here is that one of the limits is infinity. We'll handle this by treating this as a definite integral and then taking the limit as the upper limit goes to infinity. Sometimes there will be an answer, and sometimes there won't be an answer. It will basically come down to how fast the graph approaches the x-axis. If it doesn't approach the x-axis then there will be no integral for sure. If it does approach the x-axis it might. We will put some rules to this in a minute.

Example: $\int_0^\infty e^{-x}dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b$

$\lim_{b\to\infty}\int_0^b e^{-x}dx$

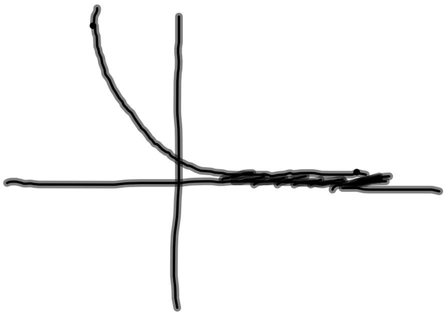
The reason this works is that $e^{-x}$ approaches zero as x→∞ and it approaches zero rapidly enough that area effectively stops accumulating.

$\lim_{b\to\infty} -e^{-x}\Big|_0^b$

$\lim_{b\to\infty} -e^{-b} - -e^{-0}$

$\lim_{b\to\infty} \overset{0}{\cancel{\frac{-1}{e^b}}} + 1$

tends to 0



$\boxed{\text{Converges}}$ $\boxed{1}$

**Definition: Improper Integrals with Infinite Integration Limits**

Integrals with infinite limits of integration are **improper integrals.**

1. If f(x) is continuous on [a, ∞), then $\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)dx$
2. If f(x) is continuous on (- ∞,b], then $\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)dx$
3. If f(x) is continuous on (- ∞, ∞), then

$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$ where c is any real number

$$\lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx$$

We say an integral **converges** if the preceding process gives a finite answer. If it doesn't, we say it **diverges.** If we have to break the integral into two parts, they both must converge, otherwise the entire integral diverges.

**Example 1:** $\int_{-\infty}^{\infty} e^x \, dx$

$$\lim_{a\to-\infty} \int_a^0 e^x dx + \lim_{b\to\infty} \int_0^b e^x dx$$

$$\lim_{a\to-\infty} e^x \Big|_a^0 + \lim_{b\to\infty} e^x \Big|_0^b$$

$$\lim_{a\to-\infty} e^0 - e^a \qquad \lim_{b\to\infty} e^b - e^0$$

$$1 - 0 \qquad e^b \to \infty$$

Since $a \to -\infty$ $e^a \to 0$

$\boxed{\text{Diverges}}$

**Example 2:** $\int_1^{\infty} \frac{dx}{x}$ Graph, then evaluate

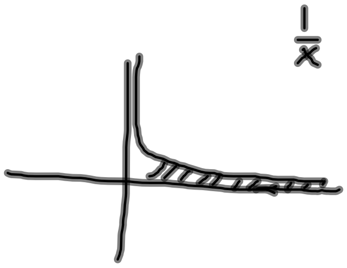
$$\lim_{b\to\infty} \int_1^b \frac{1}{x} dx$$

$$\lim_{b\to\infty} \ln|x| \Big|_1^b$$

$$\lim_{b\to\infty} \ln|b| - \ln 1$$

$$\infty - 0$$

$\boxed{\text{Diverges}}$



does not $\to 0$ fast enough

**Example 3**: $\int_0^\infty \frac{2\,dx}{x^2+4x+3}$

$$\frac{2}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$$

$$2 = A(x+1) + B(x+3)$$

Let $x=-1$ $\quad 2 = 2B \quad B=1$

Let $x=-3$ $\quad 2 = -2A \quad A=-1$

$$\lim_{b\to\infty} \int_0^b \frac{-1}{x+3} + \frac{1}{x+1}\,dx$$

$$\lim_{b\to\infty} -\ln|x+3| + \ln|x+1| \Big|_0^b$$

$$\ln|x+1| - \ln|x+3|$$

$$\lim_{b\to\infty} \ln\left|\frac{x+1}{x+3}\right| \Big|_0^b$$

$$\lim_{b\to\infty} \underbrace{\cancel{\ln\left|\frac{b+1}{b+3}\right|}}_{\text{tends to 0}} - \ln\left|\frac{0+1}{0+3}\right|$$

$$-\ln\left|\frac{1}{3}\right|$$

$$-\ln 3^{-1}$$

converges $\boxed{\ln 3}$

$$\left(\frac{b+1}{b+3}\right)\frac{\frac{1}{b}}{\frac{1}{b}}$$

$$b\to\infty \quad \frac{\frac{b}{b}+\cancel{\frac{1}{b}}^{\nearrow 0}}{\frac{b}{b}+\cancel{\frac{3}{b}}^{\nearrow 0}} \quad \frac{1}{1}$$

**Example 4**: $\int_1^\infty xe^{-x}\,dx$

Tabular

| $dy/dx$ | $\int$ |
|---|---|
| $x$ $+$ | $e^{-x}$ |
| $1$ $-$ | $-e^{-x}$ |
| $0$ | $e^{-x}$ |

$$\lim_{b\to\infty}\int_1^b xe^{-x}\,dx$$

$$x(-e^{-x}) - 1(e^{-x})$$

$$-e^{-x}(x+1)$$

$$\lim_{b\to\infty} \frac{-1}{e^x}(x+1)\Big|_1^b$$

$$\lim_{b\to\infty} \underbrace{\cancel{\frac{-1}{e^b}(b+1)}}_{\text{tends to 0}} - \frac{-1}{e^1}(1+1)$$

$$\frac{1}{e}(2)$$

$$\boxed{\text{Converges} \quad \frac{2}{e}}$$

**Example 5**: $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

$$\lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx$$

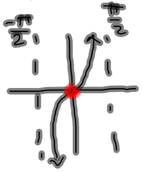
$$\lim_{a\to-\infty} \tan^{-1}(x)\Big|_a^0 \qquad \lim_{b\to\infty} \tan^{-1}(x)\Big|_0^b$$

$$\lim_{a\to-\infty} \tan^{-1}(0)-\tan^{-1}(a) \qquad \lim_{b\to\infty} \tan^{-1}(b)-\tan^{-1}(0)$$

$0 - -\frac{\pi}{2}$

$\frac{\pi}{2} - 0$

$\frac{\pi}{2}$



$\tan\theta = \text{ratio}$

$\frac{\pi}{2}+\frac{\pi}{2} = \boxed{\pi}$ $\boxed{\text{Converges}}$

**Problem 3.** $\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$

$$\lim_{a\to-\infty}\int_a^0 \frac{1}{(x^2+1)^2}\cdot 2x\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{(x^2+1)^2}\cdot 2x\,dx$$

u-sub

$u = x^2+1$

$du = 2x\,dx$

$\int \frac{1}{u^2}\,du$

$\int u^{-2}\,du$

$\frac{u^{-1}}{-1}$

$\frac{-1}{u}$

$$\lim_{a\to-\infty} \frac{-1}{x^2+1}\Big|_a^0 + \lim_{b\to\infty} \frac{-1}{x^2+1}\Big|_0^b$$

$$\lim_{a\to-\infty} \frac{-1}{0^2+1} - \cancel{\frac{-1}{a^2+1}}^{\,0} \qquad \lim_{b\to\infty} \cancel{\frac{-1}{b^2+1}}^{\,0} - \frac{-1}{0^2+1}$$

tends to 0 tends to 0

$-1$

$+1$

$-1+1$

$\boxed{\text{Converges } 0}$

The discontinuities we dealt with before were finite discontinuities. We will now look at infinite discontinuities with the same technique used for improper integrals. V.A. or hole

**Example:** $\int_0^1 \frac{dx}{\sqrt{x}}$ $[0,1]$ but $x \neq 0$ Domain $\frac{1}{\sqrt{x}}$ $(0,\infty)$

$$\lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx$$

$$\lim_{a\to 0^+} \int_a^1 x^{-\frac{1}{2}}\,dx$$

$$\lim_{a\to 0^+} 2x^{\frac{1}{2}}\Big|_a^1$$

$$\lim_{a\to 0^+} 2\sqrt{1} - \cancel{2\sqrt{a}} \xrightarrow{} 0$$

tends to 0

$\boxed{2}$

$\boxed{\text{Converges}}$

**Definition: Improper Integrals with Infinite Discontinuities**

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals**. V.A. or hole

1. If f(x) is continuous on (a, b], then $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)dx$

2. If f(x) is continuous on [a,b), then $\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)dx$

3. If f(x) is continuous on $[a,c) \cup (c,b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

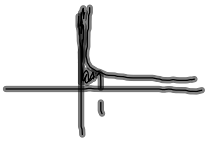
V.A is X=C

## Exploration 1:

$$\int_0^1 \frac{dx}{x^p}$$

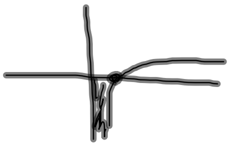
① improper because $x \neq 0$ ($\text{if } p>0$)

② $\lim\limits_{a\to 0^+} \int_a^1 \frac{1}{x}\,dx$ $\quad p=1$



$\lim\limits_{a\to 0^+} \ln|x| \Big|_a^1$

$\underset{0}{\ln 1} - \underset{-\infty}{\ln|a|}$



$\boxed{\text{diverges}}$

using # for p

Let $p=2$

③ $\lim\limits_{a\to 0^+} \int_a^1 \frac{1}{x^2}\,dx$

$\lim\limits_{a\to 0^+} \int_a^1 x^{-2}\,dx$

$\frac{x^{-1}}{-1}$

$\lim\limits_{a\to 0^+} \frac{-1}{x}\Big|_a^1$

$\lim\limits_{a\to 0^+} \frac{-1}{1} - \frac{-1}{a}$

$\boxed{\text{diverges}}$

$\int_0^1 \frac{1}{x^p}\,dx$

diverges if $p \geq 1$

④ if $0<p<1$

$\lim\limits_{a\to 0^+} \int_a^1 \frac{1}{x^{\frac{1}{2}}}\,dx$

$\int_a^1 x^{-\frac{1}{2}}\,dx$

$\lim\limits_{a\to 0^+} 2x^{\frac{1}{2}}\Big|_a^1$

$\lim\limits_{a\to 0^+} 2\sqrt{1} - \cancel{2\sqrt{a}}$ tend to 0

$\boxed{2}$ converges

$\int_0^1 \frac{1}{x^p}\,dx$

if $0<p<1$

converges

$\frac{1}{1-p}$

The integral $\int_1^{\infty} \frac{dx}{x^P}$

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx$$

$p=1$ $\quad \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$

$$\lim_{b\to\infty} \ln x \Big|_1^b$$

$$\ln b - \ln 1$$

$$\infty - 0$$

$$\boxed{\text{diverges}}$$

$p>1$

let $p=2$ $\quad \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$

$$\lim_{b\to\infty} \int_1^b x^{-2}\,dx$$

$$\frac{x^{-1}}{-1}$$

$$\lim_{b\to\infty} -\frac{1}{x}\Big|_1^b$$

$$\lim_{b\to\infty} \cancel{\frac{-1}{b}}^{\,0} - -\frac{1}{1}$$

$$+1 \qquad \boxed{\text{Converges}}\ \boxed{\frac{1}{p-1}}$$

$p<1$

Let $p=\frac{1}{2}$

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^{\frac{1}{2}}}\,dx$$

$$\lim_{b\to\infty} \int_1^b x^{-\frac{1}{2}}\,dx$$

$$\lim_{b\to\infty} 2x^{\frac{1}{2}}\Big|_1^b$$

$$\lim_{b\to\infty} 2\sqrt{b} - 2\sqrt{1}$$

$\infty$

diverges

$$\boxed{\int_1^{\infty} \frac{1}{x^p}\,dx \quad \begin{array}{ll} \text{diverges} & p\le 1 \\ \text{converges} & p>1 \quad \frac{1}{p-1} \end{array}}$$

**Example 6** $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$ $x \neq 1$

$$\lim_{b\to 1^-} \int_0^b (x-1)^{-\frac{2}{3}}\,dx + \lim_{a\to 1^+} \int_a^3 (x-1)^{-\frac{2}{3}}\,dx$$

$$\lim_{b\to 1^-} 3(x-1)^{\frac{1}{3}}\Big|_0^b \qquad 3(x-1)^{\frac{1}{3}}\Big|_a^3$$

$$\lim_{b\to 1} 3(b-1)^{\frac{1}{3}} - 3(0-1)^{\frac{1}{3}}$$

tend to 0 $+3$

$$\lim_{a\to 1^+} 3(3-1)^{\frac{1}{3}} - 3(a-1)^{\frac{1}{3}}$$

$3(2)^{\frac{1}{3}}$ tend to 0

$\boxed{\text{Converges}}$ $\boxed{3+3\sqrt[3]{2}}$

**Example 7:** $\int_1^2 \frac{dx}{x-2}$ $x \neq 2$

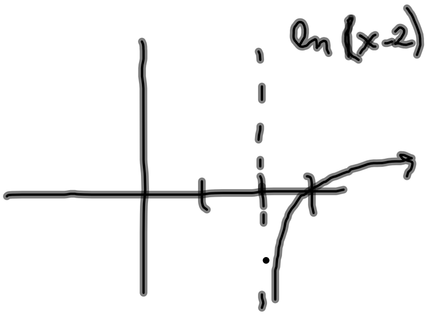
$$\lim_{b\to 2} \int_1^b \frac{1}{x-2}\,dx$$

$$\lim_{b\to 2} \ln|x-2|\Big|_1^b$$

$$\ln|b-2| - \ln|1-2|$$

$-\infty$

$\boxed{\text{diverges}}$



**25.** $\int_0^2 \frac{dx}{1-x^2}$ $\quad \frac{1}{(1-x)(1+x)}$ $\quad x \neq -1, 1$

$$\frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$1 = A(1+x) + B(1-x)$$

let $x=-1$ $\quad 1=2B$ $\quad B=\frac{1}{2}$

Let $x=1$ $\quad 1=2A$ $\quad A=\frac{1}{2}$

$$\lim_{c\to 1^-}\int_0^c \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}\,dx + \lim_{c\to 1^+}\int_c^2 \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}\,dx$$

$$\frac{1}{2}\left[\left(-\ln|1-x|\right)+\left(\ln|1+x|\right)\right]$$

$$\lim_{c\to 1^-}\frac{1}{2}\left(\ln\left|\frac{1+x}{1-x}\right|\right)\Big|_0^c + \frac{1}{2}\left(\ln\left|\frac{1+x}{1-x}\right|\right)\Big|_c^2$$

$\boxed{\text{diverges}}$

**28.** $\int_0^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$ $\quad x\neq 0$ $\quad u=\sqrt{x}=x^{\frac{1}{2}}$

$\lim_{a\to 0^+}\int_a^4 e^{-\sqrt{x}}\cdot x^{-\frac{1}{2}}dx$ $\quad du=\frac{1}{2}x^{-\frac{1}{2}}dx$

$2du=x^{-\frac{1}{2}}dx$

$$\int e^{-u}\cdot 2du$$

$$-2e^{-u}$$

$$\lim_{a\to 0^+} -2e^{-\sqrt{x}}\Big|_a^4$$

$$\lim_{a\to 0^+} -2e^{-\sqrt{4}} - -2e^{-\sqrt{a}}$$

$$\boxed{-2e^{-2}+2}$$

There are some integrals that our best tools cannot get analytic answers to

**Example 8:** $\int_1^{\infty} e^{-x^2}\,dx$ Does this integral converge?

$\lim_{b\to\infty} \int_1^b e^{-x^2}\,dx$

This function approaches the x-axis very quickly, but there is no closed form (not a nice one) that we can use in the limit. Our calculator can approximate the answer, but the best we can do analytically is to say if the integral exists.

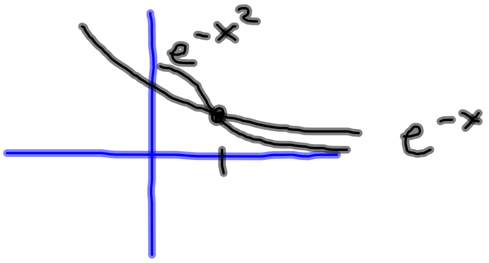
$e^{-x^2}$ $e^{-x}$

$x=1$ $e^{-1^2} = e^{-1}$

The simplest way to do this is make a comparison with a function we know. Compare $e^{-x^2}$ *with* $e^{-x}$

Both are exponentials. Both are positive and both approach zero as x→∞. Which one is larger? $e^{-x}$

Graph $e^{-x^2}$ *and* $e^{-x}$ Area must be less than $e^{-1}$. Compute using fnint$(e^{-x^2}, x, 1, x)$ $.139$



$\lim_{b\to\infty} \int_1^b e^{-x}\,dx$

$\lim_{b\to\infty} -e^{-x}\Big|_1^b$

$\lim_{b\to\infty} \frac{-1}{e^x}\Big|_1^b$

$\lim_{b\to\infty} \overset{0}{\cancel{\frac{-1}{e^b}}} - -\frac{1}{e^1}$

$\frac{1}{e}$ or $e^{-1}$

converges

$0 < e^{-x^2} \le e^{-x}$

converges

**Theorem 6: Direct Comparison Test** DCT

Let f and g be continuous on $[a, \infty)$, with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\int_a^\infty f(x)\,dx = converges\ if\ \int_a^\infty g(x)dx\ converges$
2. $\int_a^\infty g(x)\,dx = diverges\ if\ \int_a^\infty f(x)dx\ diverges$

**33.** $\int_\pi^\infty \frac{2+\cos x}{x}dx$

$$\lim_{b\to\infty}\int_\pi^b \frac{2+\cos x}{x}\,dx$$

$$-1 \le \cos x \le 1$$

add 2 $\quad +2 \quad +2 \quad +2$

÷x

$$1 \le 2+\cos x \le 3$$

$$\frac{1}{x} \le \frac{2+\cos x}{x} \le \frac{3}{x}$$

$$\lim_{b\to\infty}\int_\pi^b \frac{1}{x}\,dx$$

$$\lim_{b\to\infty} \ln x \Big|_\pi^b$$

$$\lim_{b\to\infty} \ln b - \ln\pi$$

$\boxed{\text{diverges}}$

$\therefore$ by DCT $\int_\pi^\infty \frac{2+\cos x}{x}dx$ diverges

If convergence depends on how fast f(x) →0 then, we can also determine convergence using relative rates of growth. If two functions grow at the same rate, they should both either converge or diverge together.

**Theorem 4: Limit Comparison Test**

If the positive functions f and g are continuous on $[a, \infty)$ and if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L,\ 0 < L < \infty$$

*then*

$$\int_a^\infty f(x)dx \ \ and \ \ \int_a^\infty g(x)dx$$ both converge or both diverge.

In general, knowing which comparison to use just takes practice. It is possible that both methods will work. You want to find the easiest

$$\int_1^\infty \frac{dx}{1+x^2} \qquad f(x) = \frac{1}{x^2}$$

$$g(x) = \frac{1}{1+x^2}$$

$$\lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}} \Rightarrow \frac{1}{x^2} \div \frac{1}{1+x^2}$$

$$\Rightarrow \frac{1}{x^2} \cdot \frac{1+x^2}{1}$$

$$\Rightarrow \frac{1+x^2}{x^2}$$

$$\lim_{x\to\infty} \frac{1+x^2}{x^2} \quad \frac{\infty}{\infty}$$

$$\frac{2x}{2x} \quad \frac{\infty}{\infty}$$

$$\frac{2}{2} = 1$$

by LCT since $\int_1^\infty \frac{1}{x^2}dx$ converges
integral test

$\int_1^\infty \frac{1}{1+x^2}dx$ converges

**22.** $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

$$\lim_{a\to-\infty}\int_a^0 e^{-x^2}\cdot 2x\,dx + \lim_{b\to\infty}\int_0^b e^{-x^2}\cdot 2x\,dx$$

$u = x^2$

$du = 2x\,dx$

$$\int e^{-u}\cdot du$$

$$-e^{-u}$$

$$\lim_{a\to-\infty} -e^{-x^2}\Big|_a^0 + \lim_{b\to\infty} -e^{-x^2}\Big|_0^b$$

$$\lim_{a\to-\infty}\frac{-1}{e^{x^2}}\Big|_a^0 \qquad \lim_{b\to\infty}\frac{-1}{e^{x^2}}\Big|_0^b$$

$$\lim_{a\to-\infty}\frac{-1}{e^{0^2}} - \cancel{\frac{-1}{e^{a^2}}}\to 0 \qquad \cancel{\frac{-1}{e^{b^2}}}\to 0 - \frac{-1}{e^{0^2}}$$

$$\frac{-1}{1}$$

$-1$ $\qquad +1$

$\boxed{0}$ $\boxed{\text{converges}}$

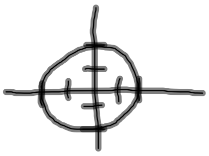
**We can use improper integrals to find some unusual things.**

**Example 9** Use the arc length to show that the circumference of the circle $x^2 + y^2 = 4$ is $4\pi$.

$C = 2\pi r$

$2\pi(2)$

$4\pi$



$y^2 = 4 - x^2$

$y = \sqrt{4-x^2}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{4-x^2}} \cdot -2x$

$\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$

$x \neq 2$

$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4-x^2}$

$$\int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\lim_{b \to 2^-} \int_0^b \sqrt{1 + \frac{x^2}{4-x^2}}\, dx$$

$$\lim_{b \to 2^-} \int_0^b \sqrt{\frac{4-x^2}{4-x^2} + \frac{x^2}{4-x^2}}$$

$$\lim_{b \to 2^-} \int_0^b \sqrt{\frac{4}{4-x^2} \cdot \left(\frac{\frac{1}{4}}{\frac{1}{4}}\right)}$$

$$\lim_{b \to 2^-} \int_0^b \sqrt{\frac{1}{1 - \frac{x^2}{4}}}$$

$$\lim_{b \to 2^-} \int_0^b \frac{1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}}\, dx$$

$\frac{d}{dx} \sin^{-1}\left(\frac{x}{2}\right)$

$\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \cdot \frac{1}{2}$

So mult by 2 to get rid of $\frac{1}{2}$

$$\lim_{b \to 2^-} 2\sin^{-1}\left(\frac{x}{2}\right)\Big|_0^b$$

$$\lim_{b \to 2^-} 2\sin^{-1}\left(\frac{b}{2}\right) - 2\sin^{-1}\left(\frac{0}{2}\right)$$

$2 \cdot \frac{\pi}{2} - 0$

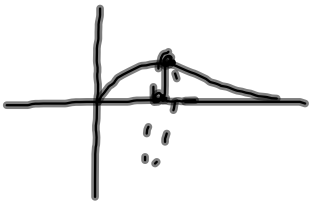
$\pi$

but that's only $\frac{1}{4}$ circumf.

Mult by 4

$\boxed{4\pi}$

**Example 10** Find the volume of the solid formed by revolving the curve $y = xe^{-x}$ about the x-axis from 0 to ∞.



A of circle
$= \pi r^2$

$r = xe^{-x}$

$$\int_0^\infty \pi(xe^{-x})^2 dx$$

$$\lim_{b\to\infty} \pi \int_0^b x^2e^{-2x}\,dx$$

Tabular

| | $\int$ |
|---|---|
| $x^2$ + | $e^{-2x}$ |
| $2x$ − | $-\frac{1}{2}e^{-2x}$ |
| $2$ + | $\frac{1}{4}e^{-2x}$ |
| $0$ | $-\frac{1}{8}e^{-2x}$ |

$$\overset{-\frac{2}{4}}{-\frac{1}{2}}x^2e^{-2x} - \frac{2}{4}xe^{-2x} + \overset{-\frac{1}{4}}{-\frac{2}{8}}e^{-2x}$$

$$\frac{1}{4}e^{-2x}(-2x^2-2x-1)$$

$$\lim_{b\to\infty} \pi\left(\frac{-2x^2-2x-1}{4e^{2x}}\Bigg|_0^b\right)$$

$$\lim_{b\to\infty} \pi\left(\cancelto{0}{\frac{-2b^2-2b-1}{4e^{2b}}} - \frac{\cancelto{0}{-2(0)^2}\cancelto{0}{-2(0)}-1}{4e^{2(0)}}\right)$$

$$\pi\left(\frac{1}{4}\right)$$

$$\boxed{\frac{\pi}{4}}$$