Notes 9.3 calculus

Relative Rates of Growth

Read page 457 through page 459 Example 2. Think about what you are reading and what it means. When you are done, write the three parts of the definition on page 457 in your own words. (What the three limits mean.)

Definitions: Faster, Slower, Same-rate Growth as x→∞

Let f(x) and g(x) be positive for x sufficiently large.

1. f grows faster than g (and g grows slower than f) as x→∞ if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty \text{ or equivalently, if } \lim_{x\to\infty}\frac{g(x)}{f(x)}=0$$

2. f and g grow at the same rate as x→∞ if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=L\neq 0 \quad \text{L finite and not zero}$$

(We can quantify the size of one in terms of the other.)

Let's look at some problems: Page 461 Determine whether the function grows faster than e^x, at the same rate as e^x, or slower than e^x as x→∞

16. 4^x

$$\lim_{x\to\infty}\frac{4^x}{e^x} \qquad \frac{4^x\cdot\ln 4}{e^x}$$

Taking L'Hopital did not help

So... $\left(\frac{4}{e}\right)^x$ $4>e$ so $\left(\frac{4}{e}\right)^x>1$

4^x grows faster.

Determine whether the function grows faster than x^2, at the same rate as x^2, or slower than x^2 as x→∞

22. 15x+3

$$\lim_{x\to\infty}\frac{15x+3}{x^2} \qquad \frac{\infty}{\infty}$$

use L'Hopital

$$\lim_{x\to\infty}\frac{15}{2x} \Rightarrow 0$$

so $15x+3$ grows slower than x^2

Determine whether the function grows faster than ln(x), at the same rate as ln(x), or slower than ln(x) as x→∞

26. $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$$\lim_{x\to\infty} \frac{x^{-\frac{1}{2}}}{\ln x}$$

OR

$$\lim_{x\to\infty} \frac{\ln x}{\frac{1}{\sqrt{x}}}$$

OR

$$\lim_{x\to\infty} \frac{\frac{1}{x^{\frac{1}{2}}}}{\ln x}$$

$$\lim_{x\to\infty} \frac{-\frac{1}{2}x^{-\frac{3}{2}}}{\frac{1}{x}}$$

$$\lim_{x\to\infty} \frac{\ln x}{x^{-1/2}}$$

$$\lim_{x\to\infty} \frac{1}{x^{\frac{1}{2}}\ln x} \to 0$$

$$\lim_{x\to\infty} \frac{x^{\frac{1}{2}}\ln x}{1} \to \infty$$

$$\frac{-1}{2x^{\frac{3}{2}}} \div \frac{1}{x}$$

$$\frac{-1}{2x^{\frac{3}{2}}} \cdot \frac{x}{1}$$

$$\frac{-1x^{1}}{2x^{\frac{3}{2}}}$$

$$\lim_{x\to\infty} \frac{-1}{2x^{\frac{1}{2}}} \to 0$$

$$\boxed{\frac{1}{\sqrt{x}} \text{ grows slower than } \ln x}$$

The transitive property of equality says that if a=b and b=c then a=c. For inequalities it says if a>b and b>c then a>c. We have something similar for rates of growth.

Transitivity of Growing Rates:
If f grows at the same rate as g as x→∞ and g grows at the same rate as h as x→∞, then f grows at the same rate as h as x→∞.

Example 5

$f(x) = \sqrt{x^2+5}$ $\quad g(x) = (2\sqrt{x}-1)^2$

Show that f(x) and g(x) grow at the same rate.

use Transitivity.

$h(x) = x$

$$\lim_{x\to\infty} \frac{f(x)}{h(x)}$$

$$\frac{\sqrt{x^2+5}}{x}$$

$$\frac{\sqrt{x^2+5}}{\sqrt{x^2}}$$

$$\sqrt{\frac{x^2+5}{x^2}}$$

$$\sqrt{\frac{x^2}{x^2}+\frac{5}{x^2}}$$

$$\lim_{x\to\infty} \sqrt{1+\cancel{\frac{5}{x^2}}}$$ tend to 0

1

grow at same rate

$$\lim_{x\to\infty} \frac{g(x)}{h(x)}$$

$$\frac{(2\sqrt{x}-1)^2}{x}$$

$$\frac{(2\sqrt{x}-1)^2}{(\sqrt{x})^2}$$

$$\left(\frac{2\sqrt{x}-1}{\sqrt{x}}\right)^2$$

$$\left(\frac{2\sqrt{x}}{\sqrt{x}}-\frac{1}{\sqrt{x}}\right)^2$$

$$\lim_{x\to\infty} \left(2-\cancel{\frac{1}{\sqrt{x}}}\right)^2$$ tends to 0

4

grow at same rate

$$\frac{f(x)}{h(x)} \div \frac{g(x)}{h(x)}$$

$$1 \div 4$$

$$\frac{f(x)}{\cancel{h(x)}} \cdot \frac{\cancel{h(x)}}{g(x)}$$

$$\frac{f(x)}{g(x)}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{1}{4}$$

$$\boxed{\frac{1}{4}}$$

grow at same rate

Read pages 460 about Sequential versus Binary Search

Example 6

$m-1 \le \log_2 n \le m$

$2^{m-1} \le n \le 2^m$

steps

$\log_2 26{,}000 \le m$

$2^m = 26{,}000$

$14.6662 \le m$

Thinking of a number between 1 and 100

$\log_2 100 \le m$

$\frac{\ln 100}{\ln 2} \le m$

$6.643 < m$