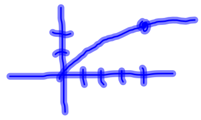
9.2 notes calculus

L'Hôpital's Rule

When we first introduced the idea of limits we used words approaches and tends to....

For example: What value does $f(x)=\sqrt{x}$ approach as x approaches 4? $\lim_{x\to 4}\sqrt{x}$ $\sqrt{4}=\boxed{2}$

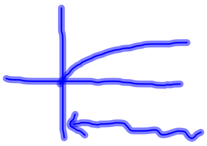


Some problems were a little trickier in that they only had a limit from one side. For example:

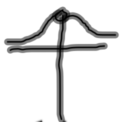
What value does $f(x)=\sqrt{x}$ approach as x approaches 0?

$\lim_{x\to 0}\sqrt{x}=?$ $\rightarrow \lim_{x\to 0^+}\sqrt{x}=0$

dne



Functions where we could just plug in the x value of interest gave us very little problem. However, other limits required other techniques. For example: when we found $\lim_{x\to 0}\frac{\sin x}{x}$



$\lim_{x\to 0}\frac{\sin 0}{0}\rightarrow\frac{0}{0}$

We resorted to a graphic approach. The problem here was that if we plugged in a zero, both the numerator and the denominator were zero or **tended** to zero. This is not a quantity we were able to compute. We could only rely on a graph.

Whenever a limit tends to the form $\frac{0}{0}$ we call that **indeterminate form**. Indeterminate forms include

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty\cdot 0,\ \infty-\infty,\ 1^{\infty},\ 0^{0},\ \infty^{0}$$

Some of these seem very strange, but we will see how to work with them.

The first tool we will have is called **L'Hôpital's Rule**. It was actually discovered by John Bernoulli, but it was popularized by Guillaume L'Hôpital because he wrote the rule in a text book. The rule is pretty simple and it is the following.

Theorem 1 L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

The idea is based on the fact that the ratio of the functions is closely related to the ratio of the derivatives. There are two clever proofs in the book. We can now analytically find a limit we have used repeatedly.

Look at Example 1

$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$ $\quad \frac{\sqrt{1+0}-1}{0} = \frac{0}{0}$ indeterminate form

use L'Hopital

$\lim_{x\to 0} \frac{f'(x)}{g'(x)} \Rightarrow \frac{\frac{1}{2\sqrt{1+x}}\cdot 1}{1} \quad \frac{\frac{1}{2\sqrt{1+0}}}{1} \quad \frac{\frac{1}{2}}{1} = \boxed{\frac{1}{2}}$

Examples: $\lim_{x\to 0}\frac{\sin(5x)}{x}$ $\qquad \lim_{x\to 1}\frac{x^3-1}{4x^3-x-3}$

$\lim_{x\to 0} \frac{\sin 5(0)}{0}$ $\quad \frac{0}{0}$ ✓ $\qquad \frac{1^3-1}{4(1)^3-1-3}$ $\quad \frac{0}{0}$ ✓

$\lim_{x\to 0} \frac{5\cos(5x)}{1}$ $\qquad \lim_{x\to 1} \frac{3x^2}{12x^2-1}$

$\frac{5\cos(5\cdot 0)}{1}$ $\qquad \frac{3(1)^2}{12(1)^2-1}$

$\boxed{5}$ $\qquad \boxed{\frac{3}{11}}$

Theorem 2: L'Hôpital's Rule (Stronger Form)
Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(a)}{g'(a)}$$

Now because we require the functions to be differentiable on an interval we can say the limit of the ratio of the functions is the same as the limit of the ratio of the derivatives. This means we can apply L'Hôpital's repeatedly until we find the limit. **BUT** each time it must be in indeterminate form.

Example 2: $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} \qquad \frac{\sqrt{1+0}-1-\frac{0}{2}}{0^2} \quad ✓\ \frac{0}{0}$$

$$\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2}}{2x} \qquad \frac{\frac{1}{2\sqrt{1+0}}-\frac{1}{2}}{2(0)} \quad ✓\ \frac{0}{0}$$

rewritten

$$\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x}$$

take derivative

$$\lim_{x\to 0}\frac{-\frac{1}{2}\cdot\frac{1}{2}(1+x)^{-\frac{3}{2}}}{2} \Rightarrow \frac{-\frac{1}{4}(1+0)^{-\frac{3}{2}}}{2} \Rightarrow \frac{-\frac{1}{4}(1)}{2}$$

$$\boxed{-\frac{1}{8}}$$

L'Hôpital's Rule can be used on one sided limits as well.

Example 3: $\displaystyle\lim_{x\to 0^+}\frac{\sin x}{x^2}$ & $\displaystyle\lim_{x\to 0^-}\frac{\sin x}{x^2}$

$$\lim_{x\to 0^+}\frac{\sin x}{x^2} \quad ✓\ \frac{0}{0} \qquad\qquad \lim_{x\to 0^-}\frac{\sin x}{x^2} \quad ✓\ \frac{0}{0}$$

$$\frac{\cos x}{2x} \quad \frac{1}{0} \quad \infty \qquad\qquad \frac{\cos x}{2x} \quad \frac{1}{0} \quad -\infty$$

not indeterminate form

* When you reach a point where one of the derivatives approaches 0 and the other does not, then the limit in question is 0 if the numerator approaches 0 or infinity if the denominator approaches 0.

$$\lim_{x\to a}\frac{0}{\#}=0 \qquad\qquad \lim_{x\to a}\frac{\#}{0}=\infty$$

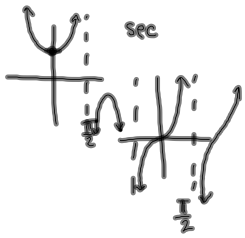
L'Hôpital's Rule applies to the forms $\frac{0}{0}$ and to the form $\frac{\infty}{\infty}$

Example: $\lim_{x\to\infty} \frac{5x^2 - 3x}{7x^2 + 1}$ $\quad \frac{\infty}{\infty}$ ✓

$\lim_{x\to\infty} \frac{10x-3}{14x} \quad \frac{\infty}{\infty}$ ✓

$\lim_{x\to\infty} \frac{10}{14} \quad \frac{10}{14} = \boxed{\frac{5}{7}}$

Example 4: $\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{1+\tan x}$



$\lim_{x\to\frac{\pi}{2}} \frac{\sec\frac{\pi}{2}}{1+\tan\frac{\pi}{2}} \quad \frac{\infty}{\infty}$

$\frac{\sec(x)\tan(x)}{\sec^2(x)}$

$\frac{\cancel{\sec x}\cdot\tan x}{\cancel{\sec x}\cdot\sec x}$

$\tan x \div \sec x$

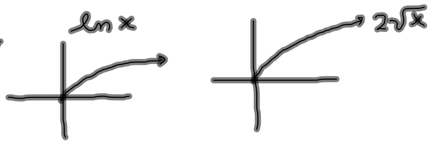
$\frac{\sin x}{\cos x} \div \frac{1}{\cos x}$

$\frac{\sin x}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{1}$

$\lim_{x\to\frac{\pi}{2}} \frac{\sin x}{1} \Rightarrow \boxed{1}$

Example 5: $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$

$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} \quad \frac{\infty}{\infty}$ ✓



$\frac{d}{dx}(2\sqrt{x})$

$\frac{\cancel{2}}{\cancel{2}\sqrt{x}}$

$\frac{d}{dx}(2(x)^{\frac{1}{2}})$

$\frac{1}{\cancel{2}}\cancel{2}(x)^{-\frac{1}{2}}$

$\frac{\frac{1}{x}}{\frac{\cancel{2}\,1}{\cancel{2}\sqrt{x}}} \Rightarrow \frac{1}{x} \div \frac{1}{\sqrt{x}}$

$\frac{1}{x} \cdot \frac{\sqrt{x}}{1}$

$\frac{x^{\frac{1}{2}}}{x^1} = x^{\frac{1}{2}-1} = x^{-\frac{1}{2}}$

$\frac{1}{\sqrt{x}}$

$\lim_{x\to\infty} \frac{1}{\sqrt{x}} \Rightarrow \boxed{0}$

All the problems so far have involved finding limits of quotients or ratios. What if we have a product? In the following problem we end up with $(\infty)(0)$ It seems the answer should be zero, but is the zero property of multiplication strong enough to cancel out a number that is infinitely large? To find out who wins, use a little trick.

Example: $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

Example: $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

Recall: Example: $\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to 0} x$

If a limit requires that we approach ∞, we can evaluate the same limit by replacing x with 1/x and approaching 0 instead.

Example: $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$ *becomes* $\lim_{x\to 0} \frac{1}{x}\sin x$ *and* $\frac{\sin x}{x} = 1$

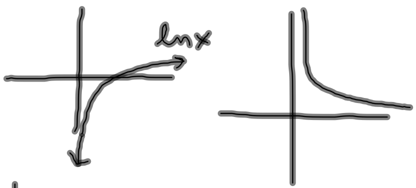
$$\frac{1}{\frac{1}{x}} \qquad 1 \div \frac{1}{x} \qquad 1 \cdot x$$

Try #17 $\lim_{x\to 0^+} x\ln(x)$

$$\lim_{x\to 0^+} x \cdot \ln(x)$$

mult by x is like $\div \frac{1}{x}$

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \qquad \frac{-\infty}{\infty}\checkmark$$



$$\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) \qquad \frac{d}{dx}(x^{-1}) \qquad -1x^{-2} \qquad \frac{-1}{x^2}$$

$$\frac{1}{x} \div \frac{-1}{x^2}$$

$$\frac{1}{x} \cdot \frac{-x^2}{1}$$

$$\lim_{x\to 0^+} -x = \boxed{0}$$

You are probably getting the idea that we somehow change things so we can use L'Hôpital's Rule. You are right. So, what if we have the indeterminate form $\infty-\infty$? By the way, what is $\infty+\infty$?

Example 7: $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

$$\frac{(x-1)}{(x-1)}\cdot\frac{1}{(\ln x)} - \frac{1}{x-1}\cdot\frac{(\ln x)}{(\ln x)}$$

$$\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)(\ln x)} \qquad \frac{0}{0}$$

$$\lim_{x\to 1}\frac{\left(\frac{x}{x}\right)1-\frac{1}{x}}{(x-1)\cdot\frac{1}{x}+(\ln x)(1)} \qquad \checkmark\ \frac{0}{0}$$

simplify CD

$$\frac{(x-1)}{x}\div\left[\frac{(x-1)}{x}+\frac{\ln x}{1}\cdot\frac{x}{x}\right]$$

$$\frac{(x-1)}{\cancel{x}}\cdot\frac{\cancel{x}}{(x-1)+x\ln x}$$

$$\lim_{x\to 1}\frac{x-1}{(x-1)+x\ln x} \qquad \frac{0}{0}$$

$$\lim_{x\to 1}\frac{1}{1+\left(x\cdot\frac{1}{x}\right)+\ln x} \Rightarrow \frac{1}{1+1+0} \quad \boxed{\frac{1}{2}}$$

Graphically this means that the vertical distance between the functions approaches half a unit as x approaches 1 from either side.

We have three more forms, but they are all fairly similar

$1^{\infty},\ 0^{0},\ \infty^{0}$

These all have exponents. What are some tools or functions that make exponents easier to deal with? log

ln

$\lim_{x\to a}\ln f(x)=L \Rightarrow \lim_{x\to a}f(x)=\lim_{x\to a}e^{\ln f(x)}=e^{L}$. Here a can be finite or infinite.

As long as the function is positive, $f(x)=e^{\ln f(x)}$

$\ln f(x) = L$

$\log_e f(x) = L$

base number power

$f(x) = e^{L}$

$e^{\ln f(x)}$

$e^{\log_e f(x)} = f(x)$

The first problem of this type that we will do has a surprising and important result. It is the following:

Example 8: $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ variable in base and as an exponent

$\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = L$ ⟸ set $= L$

$\lim\limits_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x = \ln(L)$ take the ln of both sides

put ln L on back burner

$\lim\limits_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$ power role of logs

MULT by x is same as $\div \frac{1}{x}$

$\lim\limits_{x\to\infty}\dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$ indeterminate *form* $\frac{0}{0}$

$\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}$ $\frac{0}{0}$ *replace* $\frac{1}{x}$ *with* x *and* $\lim\limits_{x\to\infty}$ *with* $\lim\limits_{x\to 0}$

$\lim\limits_{x\to 0}\dfrac{\frac{1}{1+x}}{1} = 1 = \ln(L)$ *differentiate L'Hopital*

bring back back burner

$e^1 = e^{\ln(L)}$

$e = L$

$So \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

$1 = \ln(L)$

$1 = \log_e L$

$e^1 = L$

This is a very important limit. It has a counterpart that is very easy to find. $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$

Indeterminate form 0^0

Example 9: Analyze graphically then solve.

notice x is base and an exponent

$\lim\limits_{x\to 0^+} x^x = L$ take ln of both sides

$\lim\limits_{x\to 0^+} \ln x^x = \ln L$

$\lim\limits_{x\to 0^+} x \cdot \ln x$

$\lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}}$ $\frac{\infty}{\infty}$ ✓

Take derivative

$\dfrac{\frac{1}{x}}{\frac{-1}{x^2}}$

$\frac{1}{x} \div \frac{-1}{x^2}$

$\frac{1}{x} \cdot \frac{-x^2}{1}$

$\lim\limits_{x\to 0^+} -x$

$0 = \ln L$

$0 = \log_e L$

$e^0 = L$

$\boxed{1 = L}$

Example 10: Indeterminate Form ∞^0

It seems it should be one, but ….

$$\lim_{x\to\infty} x^{\frac{1}{x}} = L$$

$$\lim_{x\to\infty} \ln x^{\frac{1}{x}} = \ln L$$

$$\lim_{x\to\infty} \frac{1}{x}\cdot \ln x$$

$$\lim_{x\to\infty} \frac{\ln x}{x} \quad \checkmark\frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$\lim_{x\to\infty} \frac{1}{x}$$

$$0 = \ln L$$

$$0 = \log_e L$$

$$e^0 = L$$

$$\boxed{1 = L}$$

Try #40

$$\lim_{x\to 0}\left(\frac{1}{x^2}\right)^x = L$$

$$\lim_{x\to 0} \ln\left(\frac{1}{x^2}\right)^x = \ln L$$

$$\lim_{x\to 0} x\cdot\ln\left(\frac{1}{x^2}\right)$$

$$\lim_{x\to 0} \frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}}$$

$$\frac{d}{dx}(x^{-2}) = -2x^{-3}$$
$$\frac{-2}{x^3}$$

take derivatives

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

$$\frac{\frac{1}{\frac{1}{x^2}}\cdot\frac{-2}{x^3}}{\frac{-1}{x^2}}$$

$$\left(1\div\frac{1}{x^2}\right)\cdot\frac{-2}{x^3}$$

numerator

$$\frac{1}{1}\cdot\frac{x^2}{1}\cdot\frac{-2}{x^3}$$

$$\frac{-2}{x}\div\frac{-1}{x^2}$$

$$\frac{-2}{x}\cdot\frac{-x^2}{1}$$

$$\frac{2x^2}{x}$$

$$\lim_{x\to 0} 2x \Rightarrow 0$$

$$0 = \ln L$$

$$e^0 = L$$

$$\boxed{1 = L}$$