9.1 notes calculus

Sequences

A **sequence** $\{a_n\}$ is a list of numbers written in an explicit order. $\{a_n\}=\{a_1, a_2, a_3, \ldots, a_n\ldots\}$ a_1 is the first term, a_2 is the second term and so forth. A sequence never ends. The numbers in the list are called the **terms** of the sequence and a_n is the **nth term**. A sequence can be finite or infinite.

Example1: Defining a Sequence Explicitly

Find the first six terms and the 100^{th} term of the sequence $\{a_n\}$ where $a_n = \frac{(-1)^n}{n^2+1}$

a_1 $n=1$ $a_1 = \frac{(-1)^1}{1^2+1} = \frac{-1}{2}$

$n=2$ $a_2 = \frac{(-1)^2}{2^2+1} = \frac{1}{5}$

$n=3$ $a_3 = \frac{(-1)^3}{3^2+1} = \frac{-1}{10}$

$n=4$ $a_4 = \frac{(-1)^4}{4^2+1} = \frac{1}{17}$

$n=5$ $a_5 = \frac{(-1)^5}{5^2+1} = \frac{-1}{26}$

$n=6$ $a_6 = \frac{(-1)^6}{6^2+1} = \frac{1}{37}$

$n=100$ $a_{100} = \frac{(-1)^{100}}{100^2+1} = \frac{1}{10001}$

The sequence in example one was defined **explicitly**.
A second way of defining a sequence is **recursively** which assigns a value to the first (or the first few) term(s) and specifies the nth term by a formula or equation that involves one or more of the terms preceding it.

Example 2: Defining a Sequence Recursively

$\boxed{b_1 = 4}$

$b_n = b_{n-1} + 2$ for all $n \geq 2$

$b_2 = b_{2-1} + 2$

$b_1 + 2$

$b_2 = 4 + 2$ $\boxed{b_2 = 6}$

$b_3 = b_2 + 2$

$6 + 2$ $b_3 = 8$

$b_4 = b_3 + 2$

$8 + 2$ $b_4 = 10$

$4, 6, 8, 10, \underline{12}, \underline{14}, \underline{16}, \boxed{\underline{18}\ b_8}$

Definition Arithmetic Sequence
A sequence $\{a_n\}$ is an **arithmetic sequence** if it can be written in the form $a_n = a_1 + (n-1)d$
$\{a, a + d, a + 2d, \ldots, \mathbf{a + (n - 1)d}, \ldots$ for some constant d. The number d is the **common difference**. Each term in an arithmetic sequence can be obtained recursively from its preceding term by adding d: $\boxed{\mathbf{a_n = a_{n-1} + d}}$ for all $n \geq 2$.

Example 3

① $-5, -2, 1, 4, 7$

$-2 - -5$

$\boxed{d = 3}$

$a_9 =$ $n=9$ $a_9 = a_1 + (n-1)d$

$a_9 = -5 + (9-1)3$

$a_9 = -5 + 24$

$\boxed{a_9 = 19}$

recursive

$a_n = a_{n-1} + d$

$\boxed{a_n = a_{n-1} + 3}$ for $n \geq 2$

Explicit: $a_n = a_1 + (n-1)d$

$a_n = -5 + (n-1)3$

$-5 + 3n - 3$

$\boxed{a_n = -8 + 3n}$

Sequence 2:

$\ln 2, \ln 6, \ln 18, \ln 54$

ⓐ $d = \ln 6 - \ln 2$ $\ln \frac{6}{2} \Rightarrow \ln 3$

ⓑ $a_9 = \ln 2 + (9-1)\ln 3$

$\boxed{a_9 = \ln 2 + 8\ln 3}$

$\ln(2 \cdot 3^8)$

c) $\boxed{a_n = a_{n-1} + \ln 3}$

d) $a_n = a_1 + (n-1)d$

$\ln 2 + (n-1)\ln 3$

$\boxed{a_n = \ln 2 + n\ln 3 - \ln 3}$

$\ln 2 + \ln 3^n - \ln 3$

$\ln 2 + \ln 3^{n-1}$

$\ln(2(3^{n-1}))$

Definition **Geometric Sequence**

A sequence $\{a_n\}$ is a **geometric sequence** if it can be written in the form $r=2$ $a_n = a_1 \cdot r^{n-1}$

$\{a, ar, ar^2, \ldots, \boxed{a \cdot r^{(n-1)}}, \ldots$ for some constant **r**. The number **r** is the **common ratio**. Each term in a geometric sequence can be obtained recursively from its preceding term by multiplying by r: $\boxed{a_n = a_{n-1} \cdot r \text{ for all } n \geq 2.}$

Example 4

Sequence 1:

$1, -2, 4, -8, 16, \ldots$

a) $\frac{-2}{1}$ $\frac{4}{-2}$ $\frac{-8}{4}$ $\Rightarrow r = -2$

b) $a_n = a_1 \cdot r^{n-1}$

$a_{10} = 1 \cdot (-2)^9$ -512

c) $a_n = a_{n-1} \cdot (-2)$

$\boxed{a_n = -2a_{n-1}}$

d) $a_n = a_1 \cdot r^{n-1}$

$a_n = 1(-2)^{n-1}$

Sequence 2:

$10^{-2}, 10^{-1}, 1, 10, 10^2$

a) $r = \frac{10}{1} = 10$

b) $a_n = a_1 \cdot r^{n-1}$

$a_{10} = 10^{-2} \cdot 10^{10-1}$

$a_{10} = 10^{-2} \cdot 10^9$

$\boxed{a_{10} = 10^7}$

recursive c) $a_n = a_{n-1}(10)$

$\boxed{a_n = 10a_{n-1}}$

explicit d) $a_n = a_1 \cdot r^{n-1}$

$a_n = 10^{-2} \cdot 10^{n-1}$

$a_n = 10^{n-1+-2}$

$\boxed{a_n = 10^{n-3}}$

Example 5

$a_2 = 6$ $a_5 = -48$

$a_n = a_1 r^{n-1}$

$a_2 = a_1 r^1$ $a_5 = a_1 r^4$

$6 = a_1 r^1$ $-48 = a_1 r^4$

$\frac{6}{r^1} = a_1$ $\frac{-48}{r^4} = a_1$

$\frac{6}{r^1} = \frac{-48}{r^4}$

$\frac{\cancel{6}r^4}{\cancel{6}} = \frac{-48r^1}{6}$

$\frac{r^4}{r^1} = \frac{-8r^1}{r^1}$

$r^3 = -8$

$\boxed{r = -2}$

$r = -2$

$6 = a_1(-2)^1$

$\frac{6}{-2} = a_1$ $\boxed{a_1 = -3}$

$a_n = a_1 \cdot r^{n-1}$

$\boxed{a_n = -3(-2)^{n-1} \text{ explicit formula}}$

Example 6:

Read use calculators

Example 7:

$b_1 = 4$ first term

$b_n = b_{n-1} + 2$ for all $n \geq 2$

recursive

$u(n) = b_n$ $u(n) = u(n-1) + 2$

$n\,Min = 1$ $u(min) = 4$

Find u using $\boxed{2nd}$ $\boxed{7}$

Definition: Limit

Let L be a real number. The sequence $\{a_n\}$ has **limit L as n approaches** ∞ if, given any positive number ϵ, there is a positive number M such that for all n > M we have
$|a_n - L| < \epsilon$
We write $\lim_{n\to\infty} a_n = L$ and say that the sequence **converges to L**. Sequences that do not have limits **diverge**.

Theorem 1: Properties of Limits

If L and M are real numbers and $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

Sum Rule: $\lim_{n\to\infty}(a_n + b_n) = L + M$ ~~Difference~~

Difference Rule: $\lim_{n\to\infty}(a_n - b_n) = L - M$

Product Rule: $\lim_{n\to\infty}(a_n b_n) = LM$

Constant Multiple Rule: $\lim_{n\to\infty}(c \cdot a_n) = c \cdot L$

Quotient Rule: $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{L}{M}$, $M \neq 0$

Example 8

$$a_n = \frac{2n-1}{n}$$

$$\lim_{n\to\infty} \frac{2n-1}{n}$$

$$\lim_{n\to\infty} \frac{2n}{n} - \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{2n}{n} - \lim_{n\to\infty} \frac{1}{n}$$

$$\lim_{n\to\infty} 2 - 0$$

Converges $L = 2$

Example 9

a) $a_n = (-1)^n \frac{n-1}{n}$

$a_1 = (-1)^1 \left(\frac{1-1}{1}\right) = 0$

$a_2 = (-1)^2 \frac{2-1}{2} = \frac{1}{2}$

$a_3 = (-1)^3 \cdot \frac{3-1}{3} = \frac{-2}{3}$

$a_4 = (-1)^4 \cdot \frac{4-1}{4} = \frac{3}{4}$

terms getting bigger

getting closer to -1 or 1

$\boxed{\text{diverges}}$

b) $b_1 = 4$

$b_n = b_{n-1} + 2$

4, 6, 8, 10

diverges

Theorem 2: The Sandwich Theorem for Sequences
If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ and if there is an integer N for which $a_n \leq b_n \leq c_n$ for all $n > N$, then $\lim_{n\to\infty} b_n = L$

Example 10 Using the Sandwich Theorem

$$a_n = \frac{\cos n}{n}$$

$$-1 \leq \cos(n) \leq 1$$

$$\frac{-1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n} \quad \text{divide by } n$$

$$\lim_{n\to\infty} \frac{-1}{n} \leq \lim_{n\to\infty} \frac{\cos(n)}{n} \leq \lim_{n\to\infty} \frac{1}{n}$$

$$0 \leq \lim_{n\to\infty} \frac{\cos(n)}{n} \leq 0$$

$L = 0$ converge

Theorem 3 Absolute Value Theorem

Consider the sequence $\{a_n\}$. If $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

If the absolute value sequence converges to 0, then the original sequence also converges to 0.