8.5 notes calculus

Applications of Integrals in Science and Statistics

In 8.1 we looked at work problems and using the formula W=Fd. We found out how much work was done. Remember that something must be moved for work to be done. Also remember that if it takes a force of 5 newtons to move an object and we move it 4 meters, we would integrate the force function from $\int_0^4 F(x)\,dx = \int_0^4 5\,dx = 5x\Big|_0^4 = 20\, newtons \cdot meter$ which is 20 joules or 20 J

Example 1: Finding the Work Done by a Force

$W = \int_0^{\frac{1}{2}} \cos \pi x$ $\qquad \frac{\sin \pi x}{\pi}\Big|_0^{\frac{1}{2}}$

$\frac{\sin \frac{\pi}{2}}{\pi} - \frac{\sin 0\pi}{\pi} \Rightarrow \boxed{\frac{1}{\pi}}$

Another Example:
Let's say we have a bucket that we are lifting off the ground. If the bucket weighs **22 newtons**, how much work is done if we lift the bucket to a height of 1 meter? 15 meters?

$W = \int_0^1 22\,dx \qquad 22x\Big|_0^1 = 22\text{ J}$

$\int_0^{15} 22\,dx \qquad 22x\Big|_0^{15} = 330\text{ J}$

Now let's say we want to draw some water from the well. We need to pull the rope up from the 15 meter well. If the rope weighs **.4 N**[illegible], how much work is done to raise the rope to the top of the well? (Assume we raised it at a constant rate.)

how far rope is raised

$$\int_0^{15} .4(15-x)dx$$

proportion of rope needing to be pulled up.

weight or Force used to raise the rope

$.4 \int_0^{15} 15 - x\,dx$

$.4\left(15x - \frac{x^2}{2}\Big|_0^{15}\right)$

$.4\left[\left(225 - \frac{225}{2}\right) - 0\right]$

45 J

So now we attach the bucket and lower it down and fill it with 50L of water or **490 newtons**. How much work is done when raising the bucket, rope and water to the top of the well? (The force needed to lift the water is equal to the water's weight.)

We can separate this into 3 problems.

a) rope $\int_0^{15} .4(15-x)dx$ 45 J

b) bucket $\int_0^{15} 22dx$ 330 J

c) water $\int_0^{15} 490dx$ 7350 J

$490x\Big|_0^{15}$

$\boxed{\text{Total work} = 7{,}725\text{ J}}$

Now, just to make sure we know how these problems work, let's say the bucket has a leak so that it was full at the bottom, but the instant it reached the top it was empty.

a) rope $\int_0^{15} .4(15-x)dx$ = 45 J

b) bucket $\int_0^{15} 22dx$ = 330 J

c) water $\int_0^{15} 490\underbrace{\left(\frac{15-x}{15}\right)}_{\text{proportion of water leaking out}}dx$

$490\int_0^{15} 1-\frac{1}{15}x \; dx$ 3675 J

$490\left(x-\frac{x^2}{30}\Big|_0^{15}\right)$

Total work 4050 J

Example 2

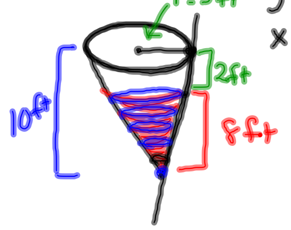
Bucket $\int_0^{20} 22\,dx = 440\text{ J}$

water $\int_0^{20} 70\left(\frac{20-x}{20}\right)dx = 700\text{ J}$

rope $\int_0^{20} .4(20-x)\,dx = 80\text{ J}$

Total work: 1220 J

Example 3



$y=2x$

$x=\frac{1}{2}y$

W= oil pumped to rim

Δ Volume = πr^2 thickness of circle

height oil is raised

$$\int_0^8 57\pi\left(\tfrac{1}{2}y\right)^2(10-y)\,dy$$

$$\int_0^8 57\pi\,\tfrac{1}{4}y^2(10-y)\,dy$$

circle πr^2

oil in tank to 8 ft

oil weight = 57 lb/ft³

$$W = \int_0^8 \frac{57\pi}{4}(y^2)(10-y)\,dy$$

Fluid Force and Fluid Pressure

If we have a 55 gallon drum, where is the pressure the greatest? If we pour the same liquid into a 15 ft diameter swimming pool, where is the pressure the greatest? How does that pressure compare to the drum? If we poured the same liquid into a 6 inch diameter cylinder that was 100 ft tall, where is the pressure the greatest? How does that pressure compare to the drum and the pool? Smaller bottom area the greater the pressure.

If you think about it, the pressure only depends on the depth, not the amount of fluid.

Fluid pressure is $\boxed{p=wh}$ where p is pressure, w is weight-density (weight per unit volume) and h is height.

Force due to pressure is $\boxed{F = p \cdot A} = w \cdot h \cdot A$ weight · height · Area

F would be in pounds, p would be in pounds/square foot and Area would be in square feet.

$F = p \cdot A$

$\text{lbs} = \frac{\text{lb}}{\cancel{\text{ft}^2}} \cdot \cancel{\text{ft}^2}$

Example 4: The Great Molasses Flood

$d = 90\text{ft}$ $r = 45$

$h = 90\text{ft}$

a) $F = pA$

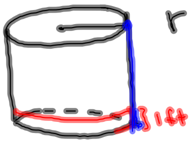
$F = w \cdot h \cdot A$

$F = 100 \cdot 90 \cdot \pi r^2$

$F = 9000\pi(45)^2$

$F \approx 57{,}255{,}526$ lbs.

ⓑ



$r = 45$

$F = w \cdot \text{height} \cdot \text{area}$

$w \cdot 2\pi r\, h$ area of band

$$\int_{89}^{90} 100\pi(2 \cdot 45)\, y\, dy$$

$$\int_{89}^{90} 9000\pi y\, dy$$

$\approx 2{,}530{,}553$ lb

Probability

Let's review some basic concepts in probability. decimals %

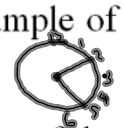
How do we usually express probabilities? fractions, ratios,

Examples: probability of rain; probability of winning at roulette or a lottery

What is the probability of a sure thing? 1 or 100%

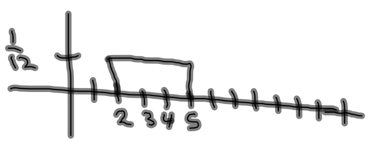
What is the probability of impossibility? 0

Knowing that all probabilities lie from 0 to 1 can be very helpful because we can construct a graphical representation. The book uses the example of the probability of a clock stopping between 2 and 5.



What portion of the clock does this cover? $\boxed{\frac{1}{4}}$ $\frac{3}{12}$

Also, since we have possibilities between 0 and 12 and they all have the same probability and the total probability is one we can construct an x, y graph.



$\frac{3}{12} = \frac{1}{4}$

How could we relate this to what we know about calculus?

$$\int_2^5 \frac{1}{12}dx \qquad \frac{1}{12}x \Big|_2^5 \qquad \frac{5}{12} - \frac{2}{12} = \frac{3}{12} = \frac{1}{4}$$

The key here was to figure out what function represented the probability.

For any situation where this can be done we call the function a PDF, Probability Density Function. In our example the PDF was 1/12.

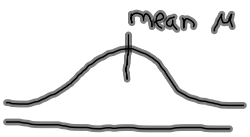
Because we know that the total probability is one we can set up an interesting integral.

$$\int_{-\infty}^{\infty} f(x)dx = 1 \qquad \int_{-\infty}^{\infty} \frac{1}{12}dx = 1$$

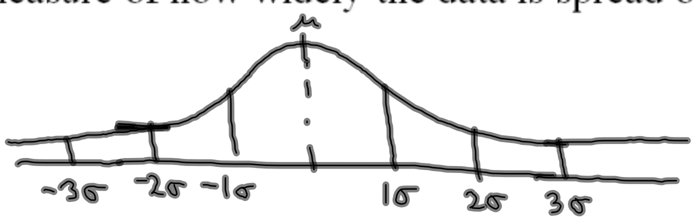
Then on any interval [a , b]

$$P = \int_a^b f(x)dx \qquad P = \int_2^5 \frac{1}{12}dx = \frac{1}{4}$$

Luckily the probability of most events follows a normal distribution (a bell curve) so we don't have to come up with a weird PDF every time. For events that have a normal distribution the PDF is actually quite easy thanks to Karl Friedrich Gause.



We only need two pieces of information from these graphs: the center, or mean $\boxed{\mu}$ and the spread, or standard deviation $\boxed{\sigma.}$ The mean is the arithmetic average. The standard deviation is a measure of how widely the data is spread or scattered.



Definition: Normal Probability Density Function (pdf) The normal probability density function (Gaussian curve) for a population with mean μ and standard deviation σ is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

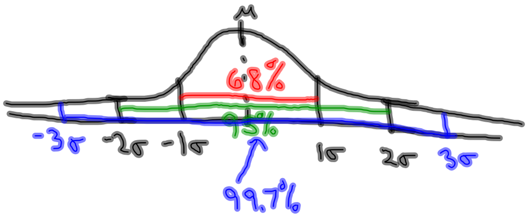
The 68-95-99.7 Rule

Given a normal curve:

68% of the area will lie within σ of the mean μ

95% of the area will lie within 2σ of the mean μ

99.7% of the area will lie within 3σ of the mean μ

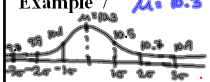


Example 6

68% from 1.5 to 2.5

99.7% [.5, 3.5]

Example 7 $\mu = 10.3$ $\sigma = 0.2$ oz



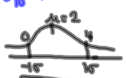
a) $\int_{10}^{11} \frac{1}{.2\sqrt{2\pi}} \cdot e^{\wedge \frac{-(x-10.3)^2}{2(.2)^2}} dx$

.93296 $\boxed{93.3\%}$

b) $\int_{9}^{10} \frac{1}{.2\sqrt{2\pi}} e^{\wedge \frac{-(x-10.3)^2}{2(.2)^2}} dx \approx .0668$ $\boxed{6.7\%}$

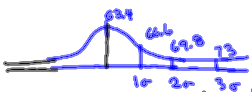
c) $\int_{10}^{10} f(x)\,dx = 0$

29.



$\boxed{68\%}$

31. $\mu = 63.4$ $\sigma = 3.2$



$4\sigma = 76.2$

$5\sigma = 79.4$

$f(x) = \frac{1}{3.2\sqrt{2\pi}} \cdot e^{\wedge \frac{-(x-63.4)^2}{2(3.2)^2}}$

a) $\int_{0}^{63.4} f(x)\,dx = 50\%$

b) $\int_{63}^{65} f(x)\,dx \approx .24120$ $\boxed{24.1\%}$

c) $h > 6$ ft

$h > 72$ in $\int_{72}^{82} f(x)\,dx \approx .003599$ $\boxed{.36\%}$

d) $\int_{60}^{60} f(x)\,dx = \boxed{0}$