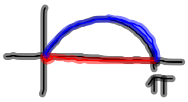
8.4 notes calculus

Length of Curves

We have seen many problems where we can see the value of finding an integral. We might use an integral to find the volume of a tank. We might find the area of an irregularly shaped field for real estate or tax purposes.

For example: the area under the sine curve from 0 to π. But, what if we wanted to fence this area in, how much fence would we need?



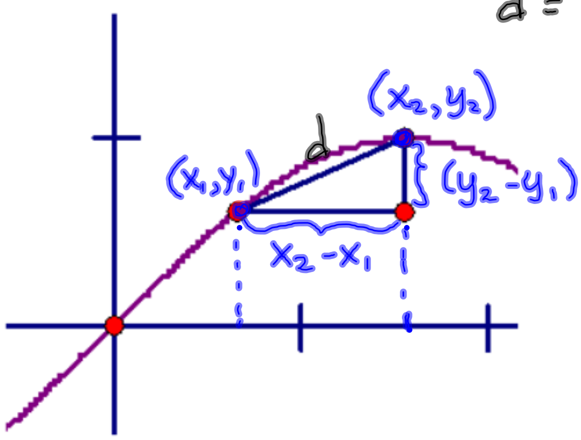
The distance from 0 to π is easy, π but what about the curve itself? The curve is $y = \sin x$

Example: What is the length of the sine curve from 0 to π?

The method goes back to the idea of "Local linearity".

Each piece of the sine curve can be made to look like a line, then using the Pythagorean Theorem or the distance formula we can compute the length of each of these pieces.

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$



$$\sqrt{(dx)^2+(dy)^2}$$

change in x   change in y

We could do this for the entire interval!

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If we sum all the little pieces we get an integral.

What are the limits? y=sinx interval was 0 to π

generally its a, b

$$\int_0^{\pi}\sqrt{(dx)^2+(dy)^2}$$

but what about dx?

You have to do some clever ~~addition~~! multiplication

$$\int_0^{\pi}\frac{\sqrt{(dx)^2+(dy)^2}}{dx}dx=\int_0^{\pi}\sqrt{\frac{(dx)^2+(dy)^2}{(dx)^2}}dx=\int_0^{\pi}\sqrt{1+\left(\frac{(dy)^2}{(dx)^2}\right)}dx$$

$$=\int_0^{\pi}\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

$$\int_a^b\sqrt{(dx)^2+(dy)^2}\cdot\frac{dx}{dx}$$

$$\int_a^b\frac{\sqrt{(dx)^2+(dy)^2}}{dx}\cdot dx$$

$$\int_a^b\frac{\sqrt{(dx)^2+(dy)^2}}{\sqrt{(dx)^2}}dx$$

$$\int_a^b\sqrt{\frac{(dx)^2+(dy)^2}{(dx)^2}}dx$$

$$\int_a^b\sqrt{\frac{(dx)^2}{(dx)^2}+\frac{(dy)^2}{(dx)^2}}dx$$

$$\boxed{L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx}$$

General formula for length of a curve

But what is dy/dx?

y= sin x and dy/dx = cos (x)

$$=\int_0^{\pi}\sqrt{1+\cos^2 x}\ dx$$

$$\int_0^{\pi}\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

This integral can't be computed analytically.

$$fnInt\approx 3.82$$

You'll notice that for this to work the first derivative must exist on the entire interval.

How does the derivative not exist?

(Cusp, corner, discontinuity or vertical tangent)

In other words, the curve must be smooth and continuous.

So if you find $\frac{dy}{dx}$ dne at any x value in [a,b] then you must do manipulation.

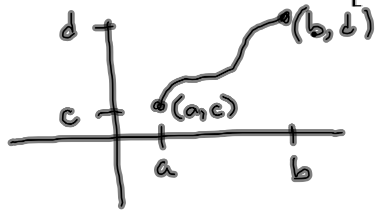
In general, this is the method.

Definition: Arc length: Length of a Smooth Curve

If a smooth curve begins at (a, c) and ends at (b, d) $a < b$, $c < d$ then the length (arc length) of the curve is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$ if y is a smooth function of x on [a , b]

y in terms of x



$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$ if x is a smooth function of y on [c , d]

x in terms of y

Once we know that a function is continuous and smooth and differentiable, we still may not end up with an easy integral. Like the last one.

However, sometimes we get lucky.

12.

$y = x^{\frac{3}{2}}$ *from* $x = 0$ *to* $x = 4$

$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$\int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx$

$\int_0^4 \left(1 + \frac{9}{4}x\right)^{\frac{1}{2}} dx$

$\int_{u(0)}^{u(4)} u^{\frac{1}{2}} \cdot \frac{4}{9}\, du$

$\frac{2}{3} \cdot \frac{4}{9} u^{\frac{3}{2}} \Big|_{u(0)}^{u(4)}$

$\frac{8}{27} u^{\frac{3}{2}} \Big|_{1}^{10}$

$\frac{8(10)^{\frac{3}{2}}}{27} - \frac{8(1)^{\frac{3}{2}}}{27}$

$\frac{8\sqrt{1000} - 8}{27}$

$\boxed{\frac{80\sqrt{10} - 8}{27}}$

$y = x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

$\left(\frac{dy}{dx}\right)^2 = \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2$

$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4}x$

$u = 1 + \frac{9}{4}x$

$\frac{du}{dx} = \frac{9}{4}$

$du = \frac{9}{4}dx$

$\frac{4}{9}du = dx$

$u(0) = 1$

$u(4) = 10$

It is possible to get around the smoothness requirement, but it can be tough.

Example 3

$y = x^{\frac{1}{3}}$ between $(-8,-2)$ $(8,2)$

a, c $\quad$ b, d

$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$

$\cancel{\int_{-8}^{8}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$ can't use this formula

$\frac{dy}{dx} = \frac{1}{3x^{2/3}}$

$\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}}$ $\quad$ $\frac{dy}{dx}$ dne at $x=0$

$y^3 = (x^{\frac{1}{3}})^3$

$y^3 = x$

$x = y^3$

$\frac{dx}{dy} = 3y^2$ $\quad$ interval $[-2,2]$

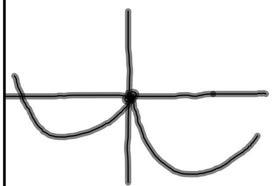
$\left(\frac{dx}{dy}\right)^2 = 9y^4$

$\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

$\int_{-2}^{2}\sqrt{1+9y^4}\,dy$

fnint ≈ 17.261

Example 4



$y = x^2 - 4|x| - x$ from $[-4,4]$

$$x^2-4|x|-x = \begin{cases} x^2+3x & \text{if } x<0 \\ x^2-5x & \text{if } x \geq 0 \end{cases}$$

$\frac{dy}{dx}$ $x<0$ $\quad 2x+3$

$\frac{dy}{dx}$ $x>0$ $\quad 2x-5$

x^2-4x-x

if x is negative drop $|\ \ |$

$x^2-4(-x)-x$

x^2+4x-x

$$\int_{-4}^{0}\sqrt{1+(2x+3)^2}\,dx + \int_{0}^{4}\sqrt{1+(2x-5)^2}\,dx$$

$\boxed{\text{fnint } 19.556}$