8.3 notes calculus

# Volumes

We are familiar with how to compute many different types of volumes: cylinders ($V = \pi r^2 h$) prisms (b·w·h), cones ($V = \frac{1}{3}\pi r^2 h$ ), and pyramids$\left(V = \frac{1}{3}Bh\right)$. We have also found volumes of other shapes using calculus. For example in chapter 6 we found the volume of a sphere by revolving a semicircle about the x axis. $y = \sqrt{16 - x^2}$ We did this using Riemann sums.

Figure 6.9 example 3 page 271.
We did this by drawing a typical cross-sectional volume which in this case was a cylinder with $r=\sqrt{16-x^2}$ and h = dx and then integrating over the domain. $\int_{-4}^{4}\pi\left(16-x^2\right)dx$

We didn't actually integrate this, the book used an MRAM program to get the answer for us; we did a similar problem and used LRAM.

We can use this same idea to get many other volumes. That is: 1) We need a volume function for a typical "cross sectional" volume. 2) We need an interval on which to integrate.

We are generally used to uniform figures, but with the help of one more idea, we can greatly expand and simplify the number of volumes we can find.

A man named Bonaventura Cavaleri came up with a theorem that says that solids with equal altitudes and identical cross section areas at each height have the same volume. As long as the cross sectional areas or values are equal, then the volumes are equal.

**Definition: volume of a solid**

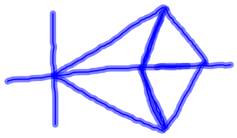
The volume of a solid of known integrable cross section area A(x) from x = a to x = b is the integral of A from a to b

$$V = \int_a^b A(x)\,dx$$

## How to find volume by the method of slicing

1. Sketch the solid and a typical cross section.
2. Find a formula for A(x)
3. Find the limits of integration
4. Integrate A(x) to find the volume.

## Example #1



Square base    height of 3

cross section is a square

$$\int_0^3 x^2 \qquad \frac{x^3}{3}\Big|_0^3 \qquad \frac{27}{3} - \frac{0^3}{3} = 9$$

$$\boxed{9\ m^3}$$

pyramid

$$\frac{1}{3}Bh$$

$$\frac{1}{3}(3\cdot3)3 = 9\ m^3$$

## Example #2

cross sections circles $\pi r^2$

$r$ = function

$r^2$ = function$^2$

$$\int_{-2}^{2} \pi r^2$$

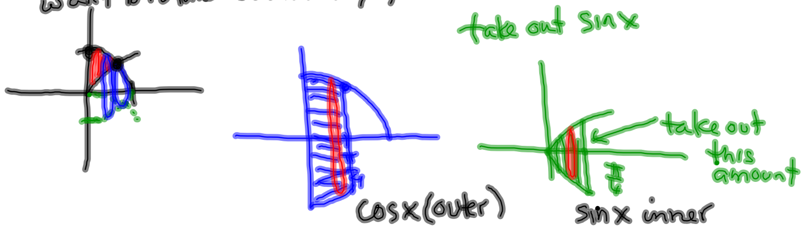
$$\pi \int_{-2}^{2} (2 + x\cos x)^2\,dx$$

52.429 cubic units

## Example #3

want to rotate bounded by y-axis, cosx and sinx

take out sinx




cross section circle $\pi r^2$

$$\int \pi r_o^2 - \pi r_i^2$$

o means outer or top function

i means inner or bottom function

$$\int \pi\cos^2 x - \int \pi\sin^2 x$$

$$\pi \int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x\ dx$$

intersection of

$\cos x = \sin x$

$x = \frac{\pi}{4}$

trig identity

$$\pi \int_0^{\frac{\pi}{4}} \cos 2x\ dx$$

$$\pi \left[ \frac{\sin 2x}{2} \Big|_0^{\frac{\pi}{4}} \right]$$

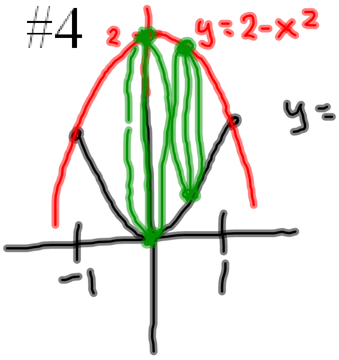
$$\pi \left[ \frac{\sin 2(\frac{\pi}{4})}{2} - \frac{\sin \cancel{2(0)}}{2} \right]$$

$$\pi \left[ \frac{1}{2} \right]$$

$$\boxed{\frac{\pi}{2} \text{ cubic units}}$$

# Problems

#4




$2-x^2=x^2$
$2=2x^2$
$1=x^2$
$\pm 1=x$

cross section circle $\pi r^2$
diameter = top to bottom
$d=2-x^2-x^2$
$d=2-2x^2$
$r=1-x^2$

$$\int_{-1}^{1} \pi r^2$$

$$\int_{-1}^{1} \pi (1-x^2)^2\, dx$$

$$\pi \int_{-1}^{1} 1-2x^2+x^4\, dx$$

$$\pi \left[ x - \frac{2x^3}{3} + \frac{x^5}{5} \Big|_{-1}^{1} \right]$$

$$\pi \left[ \left(1-\frac{2}{3}+\frac{1}{5}\right) - \left(-1-\frac{2(-1)^3}{3}+\frac{(-1)^5}{5}\right) \right]$$

$$\pi \left[ \left(\frac{15-10+3}{15}\right) - \left(-1+\frac{2}{3}-\frac{1}{5}\right) \right]$$

$$\pi \left[ \frac{8}{15} - \left(\frac{-15+10-3}{15}\right) \right]$$

$$\pi \left( \frac{8}{15} + \frac{8}{15} \right)$$

$$\boxed{\frac{16\pi}{15}}$$

# Problem 13

$y = \sqrt{9-x^2}$

$y = 0$

cross section circle $\pi r^2$

$r = \sqrt{9-x^2}$


sphere

$\frac{4}{3}\pi r^3$

$4\frac{\pi}{3} \cdot 3^3$

$4\pi \cdot 9$

$\boxed{36\pi}$

$\int_{-3}^{3} \pi r^2$

$\pi \int_{-3}^{3} \left(\sqrt{9-x^2}\right)^2 dx$

$\pi \int_{-3}^{3} 9 - x^2 \, dx$

$\pi \left[ 9x - \frac{x^3}{3} \Big|_{-3}^{3} \right]$

$\pi \left[ (27-9) - (-27 - -9) \right]$

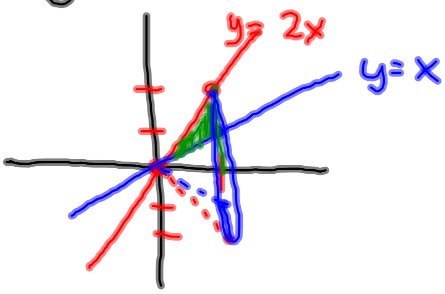
$\pi [27 - 9 + 27 - 9]$

$\pi (54 - 18)$

$36\pi$

# Problem 16

$y = 2x \quad y = x \quad x = 1$




washer method circle $\pi r^2$

outer radius inner radius

$\int_0^1 \pi(2x)^2 - \pi(x)^2 \, dx$

$\pi \int_0^1 4x^2 - x^2 \, dx$

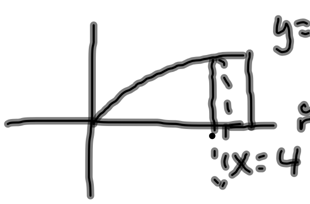
$\pi \int_0^1 3x^2 \, dx$

$\pi \left( \frac{\cancel{3}x^3}{\cancel{3}} \Big|_0^1 \right)$

$\pi [1^3 - 0^3)$

$\boxed{\pi}$

# Example 4 Finding Volumes using cylindrical shells

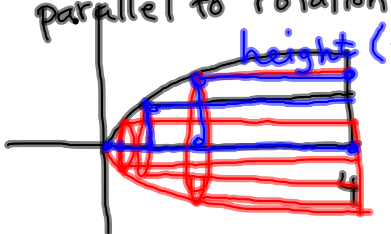



$\pi \int_0^4 (\sqrt{x})^2 dx$

cross section
circle $\pi r^2$
$r = \sqrt{x}$

$\pi \int_0^4 x\, dx$

shell method
parallel to rotation center




$\pi \left( \frac{x^2}{2} \Big|_0^4 \right) = \boxed{8\pi}$

$\pi\left(\frac{16}{2} - \frac{0^2}{2}\right)$

disc method

lateral area of cylinder

$2\pi r h$

radius = y value

thickness is dy

$2\pi\ y$ height $4-x$

$y = \sqrt{x}$

$y^2 = x$

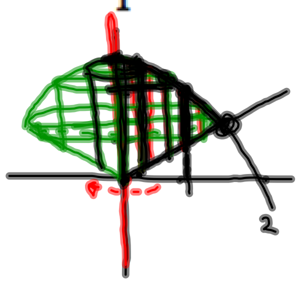
$2\pi \int_0^2 y\,(4-y^2)\, dy$

$2\pi \int_0^2 4y - y^3$

$2y^2 - \frac{y^4}{4} \Big|_0^2$

$2\pi[(8-4)-0] \Rightarrow \boxed{8\pi}$

# Example 5




$y = 4 - x^2$

$y < x$

$r = x$

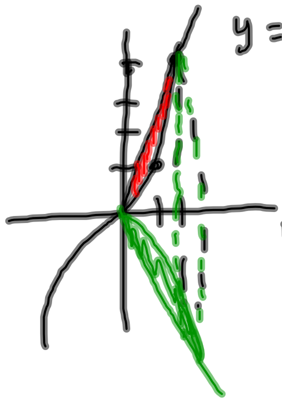
height $= 4 - x^2 - x$

$\int_0^{1.562} 2\pi r h$

$2\pi \int_0^{1.562} x(4 - x^2 - x)\, dx$

fnint $\boxed{13.327}$

Let's tackle one of the most difficult problems.
Problem 45.




$y = x^3$ $y = 4x$

$x^3 = 4x$

$x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$x = 0$ $x = \pm 2$

$$\int_0^2 \pi R_o^2 - \pi R_i^2 \, dx$$

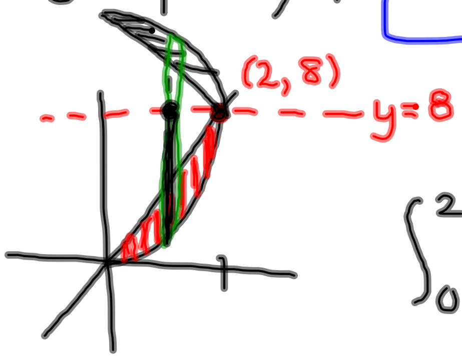
$$\int_0^2 \pi (4x)^2 - \pi (x^3)^2 \, dx$$

$$\pi \int_0^2 16x^2 - x^6 \, dx$$

$$\pi \left( \frac{16x^3}{3} - \frac{x^7}{7} \Big|_0^2 \right)$$

$$\pi \left( \frac{16(8)}{3} - \frac{2^7}{7} - 0 \right) \Rightarrow \boxed{\frac{512\pi}{21}}$$

(b)




$$\int_0^2 \pi r_o^2 - \pi r_i^2$$

outside radius $8 - x^3$

inner radius $8 - 4x$

circles $\pi r^2$

$$\pi \int_0^2 (8 - x^3)^2 - (8 - 4x)^2 \, dx$$

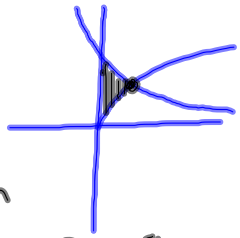
fnint $\boxed{\frac{832\pi}{21}}$

Quick Quiz 4a.4b.4c.

p415

$y_1 = \sqrt{x}$

$y_2 = e^{-x}$



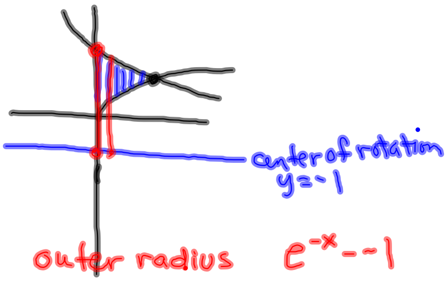
intersection

$K \approx .4263028$

a) $\int_0^K e^{-x} - \sqrt{x}\, dx$

fnint .16152

b)




outer radius $e^{-x} - -1$ or $e^{-x}+1$

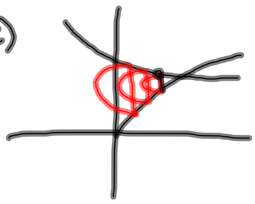
inner radius $\sqrt{x} - -1$ or $\sqrt{x}+1$

$\int \pi r_o^2 - \pi r_i^2\, dx$

$\pi \int_0^K (e^{-x}+1)^2 - (\sqrt{x}+1)^2\, dx$

fnint $\boxed{1.631}$

c)



$A_{semicircle} = \frac{1}{2}\pi r^2$

diameter $= e^{-x} - \sqrt{x}$

radius $= \dfrac{e^{-x} - \sqrt{x}}{2}$

$\int \frac{1}{2}\pi r^2$

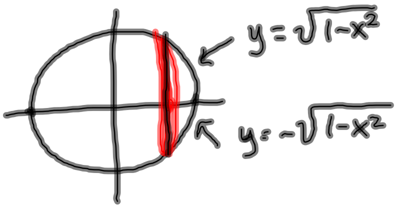
$\frac{1}{2}\pi \int_0^K \left(\dfrac{e^{-x} - \sqrt{x}}{2}\right)^2 dx$

$\frac{1}{8}\pi \int_0^K (e^{-x} - \sqrt{x})^2$

fnint $\boxed{.03476}$

Unfortunately, there is no substitute for practice.

Let's do #1 together.
Find a formula for the area A(x) of the cross sections of the solid that are perpendicular to the x-axis. You do not have to compute the volume.




circle $\pi r^2$

$d=\sqrt{1-x^2}--\sqrt{1-x^2}$

$d=2\sqrt{1-x^2}$

$r=\sqrt{1-x^2}$

$A(x)=\pi\left(\sqrt{1-x^2}\right)^2$

$\boxed{A(x)=\pi(1-x^2)}$

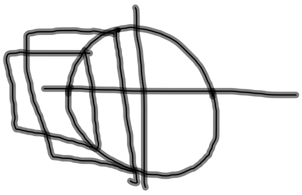
b square side²

$A=\text{side}^2$

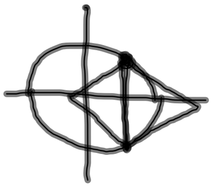
side = diameter

$A(x)=\left(2\sqrt{1-x^2}\right)^2$

$\boxed{4(1-x^2)}$



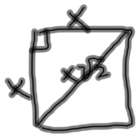
c)



diagonal = side $\sqrt{2}$

diagonal = diameter

$\dfrac{2\sqrt{1-x^2}}{\sqrt{2}}=\dfrac{\text{side}\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}$



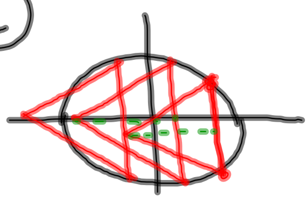

$A(x)=\text{side}^2$

$A(x)=\left(\dfrac{2\sqrt{1-x^2}}{\sqrt{2}}\right)^2$

$A(x)=\dfrac{\overset{2}{\cancel{4}}(1-x^2)}{\cancel{2}}$

$\boxed{2(1-x^2)}$

d



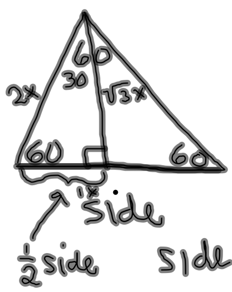
base = diameter

$A=\frac{1}{2}\,\text{base height}$

$\frac{1}{2}\left(2\sqrt{1-x^2}\right)\left(\sqrt{1-x^2}\right)\left(\sqrt{3}\right)$

$\sqrt{1-x^2}\cdot\sqrt{1-x^2}\cdot\sqrt{3}$

$\boxed{\sqrt{3}(1-x^2)}$




$\text{side}=2\sqrt{1-x^2}$