8.2 notes calculus

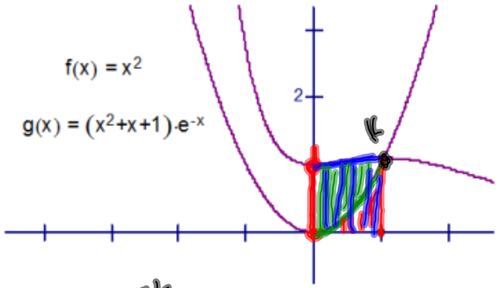
Areas in the Plane

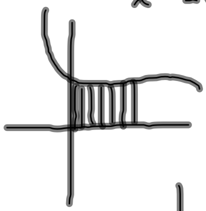
Find the area of the region enclosed by the y-axis and the curves $y=x^2$ *and* $y=(x^2+x+1)e^{-x}$

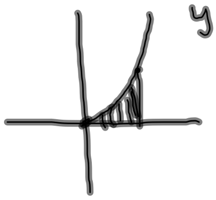
Draw the graph.

Find k - Find intersection

$x^2=(x^2+x+1)e^{-x}$







$$\int_0^k g(x) - f(x)\ dx$$

$$\int_0^k (x^2+x+1)e^{-x} - x^2\ dx$$

$k = 1.0503204$

This is an area so we know we will integrate. But what are the limits? In which direction should we integrate?

$\int_0^k dx$ Start at zero, end at intersection and integrate with respect to x.

How tall is each of the rectangles?

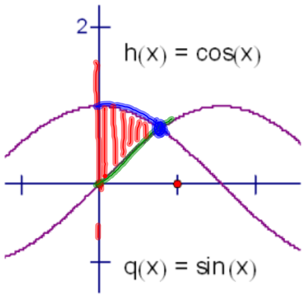
Top function – bottom function

$$\int_0^k \left(\left(x^2+x-1\right)e^{-x}-x^2\right)dx$$

This is nasty. Let's apply this idea to easier functions.

Example: Find the area bound by the y-axis.
y = cos (x) and y = sin(x)

$$\int_0^k (\cos x - \sin x)\,dx$$



K is intersection cos x = sin x

$K = \frac{\pi}{4}$

$$\int_0^{\frac{\pi}{4}} \cos x - \sin x \; dx$$

$$\sin x - -\cos x \Big|_0^{\frac{\pi}{4}}$$

$$\sin x + \cos x \Big|_0^{\pi/4}$$

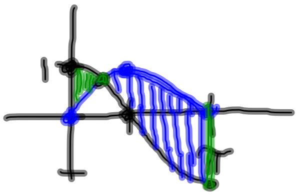
$$\left(\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right) - (\sin 0 + \cos 0)$$

$$\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} \quad - \quad (0 + 1)$$

$$\boxed{\sqrt{2} - 1}$$

Definition: **Area Between Curves**: If f and g are continuous with f(x) ≥ g(x) throughout [a,b], then the **area between the curves y = f(x) and y = g(x) from a to b** is the integral of [f – g] from a to b. $A = \int_a^b [f(x) - g(x)]dx$

Example: Area bounded by cos (x) and sin (x) on [0,π]
Graph cos (x) and sin(x)
Which function is on top? both
Is there a problem stopping at π?
yes!
both functions are on top at different intervals



$$\int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\frac{\pi}{4}}^{\pi} \sin x - \cos x \, dx$$

We split this into two integrals. $\left[0, \frac{\pi}{4}\right]$ with cos(x) on top and $\left[\frac{\pi}{4}, \pi\right]$ with sin(x) on top.

$$A=\int_0^{\frac{\pi}{4}}(\cos x-\sin x)\,dx+\int_{\frac{\pi}{4}}^{\pi}(\sin x-\cos x)\,dx$$

$$= \sin x+\cos x\Big|_0^{\frac{\pi}{4}} + -\cos x-\sin x\Big|_{\frac{\pi}{4}}^{\pi}$$

$$= \left(\sin\tfrac{\pi}{4}+\cos\tfrac{\pi}{4}\right)-(\sin 0+\cos 0)+\left[\left(\overset{-(-1)}{-\cos\pi}-\sin\pi\right)-\left(-\cos\tfrac{\pi}{4}-\sin\tfrac{\pi}{4}\right)\right]$$

$$= (\sqrt{2}-1)+\left[(1-0)-\left(-\tfrac{\sqrt{2}}{2}-\tfrac{\sqrt{2}}{2}\right)\right]$$

$$\sqrt{2}-1+1+\sqrt{2}$$

$$=\boxed{2\sqrt{2}}$$

Now let's take a look at another problem with an easier area, but some interesting alternatives.

Example: Area bounded by x-axis, $y=\sqrt{x}$ and $y=x-2$

Graph the function.

Intersection

$\sqrt{x}=x-2$

$x=(x-2)^2$

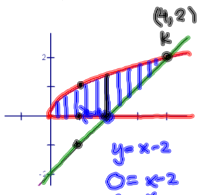
$x=x^2-4x+4$

$0=x^2-5x+4$

$0=(x-4)(x-1)$

$x-4=0 \quad x-1=0$

$x=4 \quad \cancel{x=1}$ Extraneous



$y=x-2$

$0=x-2$

$2=x$

We split the integral into two pieces with $\sqrt{x}$ on top, x-axis below from (0,2)

$\sqrt{x}$ on top and x-2 below from (2,4)

$$\int_0^2\sqrt{x}\,dx+\int_2^4\sqrt{x}-(x-2)\,dx$$

$$\int_0^2 x^{\frac{1}{2}}dx+\int_2^4 x^{\frac{1}{2}}-x+2\,dx$$

$$\frac{2}{3}x^{\frac{3}{2}}\Big|_0^2 \qquad \frac{2}{3}x^{\frac{3}{2}}-\frac{x^2}{2}+2x\Big|_2^4$$

$$\frac{2}{3}(2)^{\frac{3}{2}}-\frac{2}{3}(0)^{\frac{3}{2}} \qquad \left[\frac{2}{3}(4)^{\frac{3}{2}}-\frac{4^2}{2}+2(4)\right]-\left[\frac{2}{3}(2)^{\frac{3}{2}}-\frac{2^2}{2}+2(2)\right]$$

$$\frac{2}{3}\sqrt{8}-0 \qquad \left(\frac{16}{3}\cancel{-8+8}\right)-\left(\frac{4\sqrt{2}}{3}\underbrace{-2+4}_{2}\right)$$

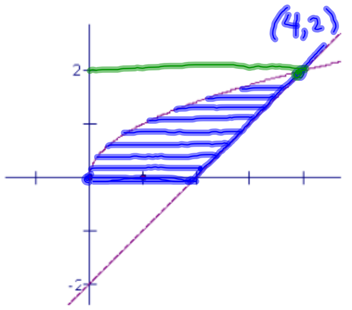
$$\cancel{\frac{4\sqrt{2}}{3}}+\frac{16}{3}\cancel{-\frac{4\sqrt{2}}{3}}-2$$

$$\frac{16}{3}-2\Rightarrow\frac{16}{3}-\frac{6}{3}\Rightarrow\boxed{\frac{10}{3}}$$

Because there is no single function on the bottom, we had to break it into 2 separate integrands.

We have always been used to integrating with respect to x because the width of our Riemann rectangles was always horizontal. What if we ran our rectangles the other way? What if we looked at it like this.

dy right function – left function



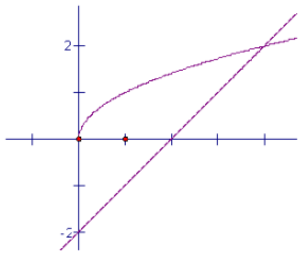
$$\int_0^2 (x-2) - \sqrt{x} \ dy$$

$$\int_0^2 (y+2) - y^2 \ dy$$

Is there a problem with saying right function minus left function? We are integrating with respect to y, each function needs to be written in terms of y.

$y = x-2$ $\quad$ $y = \sqrt{x}$

$y+2 = x$ $\quad$ $y^2 = x$



x = y + 2 $\quad$ x = y^2

$$\int_0^2 right - left \ \ dy$$

$$\int_0^2 y + 2 - y^2 \ \ dy$$

$$\frac{y^2}{2} + 2y - \frac{y^3}{3} \Big|_0^2$$

$$\left(\frac{2^2}{2} + 2(2) - \frac{2^3}{3}\right) - (0)$$

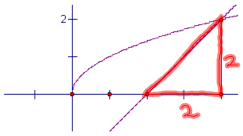
$$2 + 4 - \frac{8}{3}$$

$$6 - \frac{8}{3} \Rightarrow \frac{18-8}{3} = \frac{10}{3}$$

So, we can make this problem one single integral by integrating with respect to y rather than x. This is a good time-saving trick to have.

Another trick that is very clever involves finding integrals with geometric formulas.

Did anyone notice a shape in the area each time we graphed it?



Area of triangle

$\frac{1}{2}(2)(2) = 2$

This triangle has area 2 or $\int_0^4 \sqrt{x}\,dx - 2$ OR $\sqrt{x}$ isn't very nice to integrate

$$\frac{2}{3}x^{\frac{3}{2}}\Big|_0^4 \quad -2$$

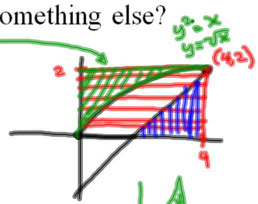
$$\frac{2}{3}(4)^{\frac{3}{2}} - \cancel{\frac{2}{3}(0)^{\frac{3}{2}}} \quad -2$$

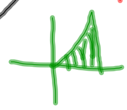
$$\frac{16}{3} - 2 = \boxed{\frac{10}{3}}$$

Can anyone think of something else?

$$8 - 2 - \int_0^2 y^2\,dy$$

rectangle − triangle −





$$6 - \frac{y^3}{3}\Big|_0^2$$

$$6 - \frac{8}{3}$$

$$\frac{18-8}{3} = \frac{10}{3}$$

The moral of these examples is that you are not always limited to the type of integral you set up. dy, dx, subregions and even geometry formulas can be useful.

Let's discuss options on problems 1 – 4. Let's choose one problem to be done together.

① $\int_0^{\pi} 1 - \cos^2 x$

$\int \cos^2 x\,dx$?

$\int \sin^2 x\,dx$?

$$\int_0^{\pi} 1 - \left(\frac{1+\cos 2x}{2}\right) dx$$

$\cos(2x) = 2\cos^2 x - 1$

$$\boxed{\frac{\cos(2x)+1}{2} = \boxed{\cos^2 x}}$$

$$\int_0^{\pi} 1 - \frac{1}{2} - \frac{\cos 2x}{2}\,dx$$

$\cos 2x = 1 - 2\sin^2 x$

$$\int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos 2x\,dx$$

$\frac{\cos 2x - 1}{-2} = \sin^2 x$

$$\frac{1}{2}\int_0^{\pi} 1 - \cos 2x\,dx$$

$$\boxed{\frac{1-\cos 2x}{2} = \sin^2 x}$$

$$\frac{1}{2}\left[x - \frac{\sin 2x}{2}\Big|_0^{\pi}\right]$$

$\int \cos kx\,dx$

$\frac{\sin kx}{k}$

$$\frac{1}{2}\left[\left(\pi - \cancel{\frac{\sin(2\pi)}{2}}^{\,0}\right) - \cancel{\left(0 - \frac{\sin(2\cdot 0)}{2}\right)}^{\,0}\right]$$

$\frac{1}{2}\pi$ or $\frac{\pi}{2}$

2) $2\int_0^{\frac{\pi}{3}} \frac{1}{2}\sec^2 t - -4\sin^2 t \; dt$

$\int_0^{\frac{\pi}{3}} \sec^2 t + 8\sin^2 t \; dt$

$\sin^2 t = \frac{1-\cos 2t}{2}$

$\int_0^{\frac{\pi}{3}} \sec^2 t + \cancel{8}^{4}\left(\frac{1-\cos 2t}{\cancel{2}}\right) dt$

$\int_0^{\frac{\pi}{3}} \sec^2 t + 4 - 4\cos 2t \; dt$

$\tan t + 4t - \frac{\cancel{4}^{2}\sin 2t}{\cancel{2}} \Big|_0^{\frac{\pi}{3}}$

$\left[\tan\frac{\pi}{3} + 4\left(\frac{\pi}{3}\right) - 2\sin\left(\frac{2\pi}{3}\right)\right] - \cancel{\left[\tan 0 + 4(0) - 2\sin 0\right]}$ 0

$\tan\frac{\pi}{3} + \frac{4\pi}{3} - \cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)$

$\sqrt{3} + \frac{4\pi}{3} - \sqrt{3}$

$\boxed{\frac{4\pi}{3}}$

4) $\int_0^1 (12y^2 - 12y^3) - (2y^2 - 2y) \; dy$

$\int_0^1 12y^2 - 12y^3 - 2y^2 + 2y \; dy$

$\frac{12y^3}{3} - \frac{12y^4}{4} - \frac{2y^3}{3} + \frac{2y^2}{2}$

$4y^3 - 3y^4 - \frac{2y^3}{3} + y^2 \Big|_0^1$

$\left(4 - 3 - \frac{2}{3} + 1\right) - (0)$

$2 - \frac{2}{3}$

$\frac{6-2}{3} = \boxed{\frac{4}{3}}$