7.5 notes calculus

Logistic Growth

Before we begin the logistic growth rates and equations we need to review partial fraction decomposition.

Definition: Partial Fraction Decomposition with Distinct Linear Denominators

If $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials with the degree of P **less than** the degree of Q, and if Q(x) can be written as a product of distinct linear factors, then f(x) can be written as a sum of rational functions with distinct linear denominators.

Given a fraction like $\frac{5x-1}{x^2+x-12}$ *it can be written as* $\frac{3}{x+4}+\frac{2}{x-3}$ this is referred to as partial fraction decomposition and the two simpler fractions are called partial fractions. $\frac{P}{Q}$ is called proper if the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator. Otherwise, the rational expression is termed improper. The partial fraction decomposition of the rational expression $\frac{P}{Q}$ depends on the factors of the denominator Q. The denominator Q will contain only factors of one or both of the following types: **Linear factors of the form x-*a*,** where a is a real number or irreducible **quadratic factors** of the form ax^2+bx+c, where a, b, and c are real numbers and $a \neq 0$ and $b^2 - 4ac < 0$. There are four cases, but we will deal with two.

Case 1: Q has only <u>nonrepeated linear</u> factors.

$$\frac{P(x)}{Q(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\ldots+\frac{A_n}{x-a_n}$$

Case 2: Q has <u>repeated linear</u> factors.

$$\frac{P(x)}{Q(x)}=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\ldots+\frac{A_n}{(x-a)^n}$$

Example 1

$$\frac{x-13}{2x^2-7x+3}$$

$$\frac{x-13}{(2x-1)(x-3)} = \frac{A}{(2x-1)}\cdot\frac{(x-3)}{(x-3)} + \frac{B}{x-3}\cdot\frac{(2x-1)}{(2x-1)}$$

common denom

Denoms are now same so....
Just worry about numerators.

$$x-13 = A(x-3)+B(2x-1)$$

Linear substitution   Let $x=3$

$$3-13 = \cancel{A(3-3)} + B(2(3)-1)$$

$$-10 = 5B$$

$$\boxed{-2 = B}$$

Let $x=\frac{1}{2}$

$$\frac{1}{2}-13 = A\left(\frac{1}{2}-3\right) + \cancel{B\left(2\cdot\frac{1}{2}-1\right)}$$

$$-\frac{2}{5}\cdot\frac{-25}{2} = -\frac{5}{2}A\cdot -\frac{2}{5}$$

-5

$$\boxed{5 = A}$$

$$\boxed{\frac{5}{2x-1} + \frac{-2}{x-3}}$$

Example 2

$$\int \frac{3x^4+1}{x^2-1}\,dx$$

improper
do long division

D $\div$
M $*$
S sub
B ring down

$$\begin{array}{r} 3x^2+3+\frac{4}{x^2-1} \\ x^2-1\,\overline{)\,3x^4+0x^3+0x^2+0x^1+1} \\ -\left(\cancel{3x^4}+3x^2\right) \\ \cancel{3x^2}+0x+1 \\ -\left(\cancel{3x^2}+3\right) \\ \hline 4 \end{array}$$

$\boxed{3x^2}(x^2-1)$

$\boxed{3}(x^2-1)$

4 remainder

$$\int 3x^2+3+\frac{4}{x^2-1}\,dx$$

$$\frac{4}{\underset{(x+1)(x-1)}{x^2-1}} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$4 = A(x-1)+B(x+1)$$

Let $x=1$ $\quad 4 = \cancel{A(1-1)} + B(1+1)$

$4 = 2B \quad \boxed{B=2}$

Let $x=-1$ $\quad 4 = A(-1-1)$

$4 = -2A \quad A=-2$

$$\int 3x^2+3+\frac{-2}{x+1}+\frac{2}{x-1}+C$$

$$x^3+3x+-2\ln|x+1|+2\ln|x-1|+C$$

$$x^3+3x+2\left(\ln|x-1|-\ln|x+1|\right)+C$$

$$x^3+3x+2\ln\left|\frac{x-1}{x+1}\right|+C$$

Example 3

$$\frac{dy}{dx} = \frac{6x^2-8x-4}{(x^2-4)(x-1)}$$

$$\int dy = \int \frac{6x^2-8x-4}{(x-2)(x+2)(x-1)}\,dx$$

$$\frac{6x^2-8x-4}{(x-2)(x+2)(x-1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x-1}$$

$$6x^2-8x-4 = A(x+2)(x-1) + B(x-2)(x-1) + C(x-2)(x+2)$$

Let $x=1$ $\quad 6-8-4 = C(1-2)(1+2)$

$$-6 = C(-3)$$

$$2 = C$$

Let $x=-2$ $\quad 6(-2)^2-8(-2)-4 = B(-2-2)(-2-1)$

$$24+16-4 = 12B$$

$$36 = 12B$$

$$3 = B$$

Let $x=2$ $\quad 6(2)^2-8(2)-4 = A(2+2)(2-1)$

$$24-16-4 = 4A$$

$$4 = 4A$$

$$1 = A$$

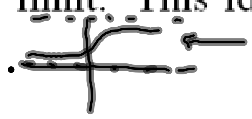
$$\int \frac{1}{x-2} + \frac{3}{x+2} + \frac{2}{x-1}\,dx$$

$$\ln|x-2| + 3\ln|x+2| + 2\ln|x-1| + C$$

$$\ln|x-2| + \ln|x+2|^3 + \ln|x-1|^2 + C$$

$$\ln\left(|x-2|\cdot|x+2|^3\cdot|x-1|^2\right) + C$$

In the previous section we used exponential equations to represent growth. These equations came from differential equations. Recall the slope of an exponential graph. Does such a graph work well for money? Why? Does such a graph work well for population? Why? Exponential equations do work well, but a better fit would be an equation that takes into consideration an upper limit. This idea gives rise to the idea of logistic growth equations.



The general Logistic Formula: The solution of the general logistic differential equation $\frac{dP}{dt} = kP(M-P)$ is $P = \frac{M}{1+Ae^{-(k)t}}$ where A is a constant determined by an appropriate initial condition. The carrying capacity M and the growth constant k are positive constants. This book uses $P = \frac{M}{1+Ae^{-(Mk)t}}$ on some of its problems. For the AP exam you need to recognize all 3 forms of the logistics differential equation: $\frac{dP}{dt} = kP(M-P)$; $\frac{dP}{dt} = kP\left(1-\frac{P}{M}\right)$; $\frac{dP}{dt} = \frac{kP}{M}(M-P)$. Any of the three forms might appear on the AP exam (and have appeared!).

Notice that as P approaches M the growth rate decreases and if P got bigger than M the rate would be negative and the population would then decrease. P is current population, t is time, M is the carrying capacity of the environment, and Mk is a positive proportionality constant. A is found using the initial population P(0). When finding the regression formula and when solving the differential equation the general logistic formula is $P = \frac{M}{1+Ae^{-kt}}$

2 ☆ logistic formula

2 Horizontal Asymptotes
(upper carrying capacity)

$\frac{1}{2}$ carrying capacity is where the rate is increasing the fastest

Now let's do some problems.

13. $\displaystyle\int \frac{8x-7}{2x^2-x-3}\,dx$

$(2x-3)(x+1)$

$$\frac{8x-7}{(2x-3)(x+1)} = \frac{A}{(2x-3)} + \frac{B}{(x+1)}$$

$$8x-7 = A(x+1) + B(2x-3)$$

Let $x = -1$ $\quad -15 = B(2(-1)-3)$

$$-15 = -5B$$

$$3 = B$$

Let $x = \frac{3}{2}$ $\quad 8\left(\frac{3}{2}\right) - 7 = A\left(\frac{3}{2}+1\right) + \cancel{B\left(2\cdot\frac{3}{2}-3\right)}$

$$12-7 = A\left(\frac{5}{2}\right)$$

$$\frac{2}{\cancel{5}}\cdot\cancel{5} = \cancel{\frac{5}{2}}A\cdot\cancel{\frac{2}{5}}$$

$$2 = A$$

$$\int \frac{2}{2x-3} + \frac{3}{x+1}\,dx$$

$$\frac{\cancel{2}\ln|2x-3|}{2} + 3\ln|x+1| + C$$

chain rule

$$\ln|2x-3| + 3\ln|x+1| + C$$

"BOB" $\quad \boxed{\ln\left(|2x-3|\cdot|x+1|^3\right) + C}$

25.

$$\frac{dP}{dt} = .0002P(1200-P)$$

$$\frac{dP}{dt} = kP(M-P)$$

a) carrying capacity = 1200

b) size of population when its growing the fastest.

$\boxed{600}$

c) $\frac{dP}{dt}(600) = .0002(600)(1200-600)$

72 people per year

31.

$$P(t) = \frac{1000}{1+e^{4.8-.7t}}$$

a) $$\frac{1000}{1+e^{4.8-.7t}} = \frac{M}{1+Ae^{-Mkt}}$$

$M = 1000$

$A = e^{4.8}$

$k = .00007$

$\cancel{1} + e^{4.8-.7t} = \cancel{1} + Ae^{-Mkt}$

$e^{4.8} \cdot e^{-.7t} = A e^{-Mkt}$

$e^{-.7t} = e^{-Mkt}$

$-.7t = -Mkt$

$-.7 = -Mk$

$-.7 = -1000k$

$\frac{-.7}{-1000} = k$

b) $t=0$

$$P(0) = \frac{1000}{1+e^{4.8-.7(0)}}$$

$$P(0) = \frac{1000}{1+e^{4.8-0}}$$

$$P(0) = \frac{1000}{1+e^{4.8}}$$

≈ 8 rabbits