7.4 notes

Exponential Growth and Decay

What is the advantage of putting money in the bank or in a stock fund or mutual fund? When we earn interest, how is it computed? If you put $100 in at 6%, how much do you get? How often an account pays interest is when it compounds the interest, but the amount that interest is paid is changing all the time. The rate of change depends on how much is in the account, so now we can set up an equation.
The first thing we need to do is understand what a separable differential equation is.

Definition: Separable Differential Equation: A differential equation of the form $\frac{dy}{dx} = f(y)g(x)$ is called **separable.**
We **separate the variables** by writing it in the form $\frac{1}{f(y)}dy = g(x)dx$. The solution is found by antidifferentiating each side with respect to its thusly isolated variable.

Example 1:

$$\frac{dy}{dx} = (xy)^2 \quad \text{when } x=1\ y=1$$

$$\frac{dy}{dx} = x^2y^2$$

$$\int \frac{dy}{y^2} = \int x^2\,dx$$

$$\int y^{-2}dy = \frac{x^3}{3} + C$$

$$\frac{y^{-2+1}}{-2+1}$$

$$-1y^{-1} = \frac{x^3}{3} + C$$

$$-\frac{1}{y} = \frac{x^3}{3} + C$$

$$-\frac{1}{1} = \frac{1^3}{3} + C$$

$$-1 - \tfrac{1}{3} = C \qquad C = -\tfrac{4}{3}$$

$$\frac{-1}{y} = \frac{x^3}{3} + -\frac{4}{3}$$

$$\frac{-1}{y} = \frac{x^3-4}{3}$$

$$-3 = (x^3-4)y$$

$$\frac{3}{4-x^3} = y \quad \text{or} \quad \boxed{\frac{-3}{x^3-4} = y}$$

p 355 Read

Exponential Change

$$\frac{dy}{dt} = ky$$

k = growth or decay constant

$$\int \frac{1}{y}\,dy = \int k\,dt$$

separate
mult by dt
÷ by y

$$\ln|y| = kt + C$$

integrate
antidifferentiate

$\log_e|y| = kt + c$

base number power

$\text{base}^{\text{power}} = \text{number}$

$$|y| = e^{kt+C}$$

$$|y| = e^{C}e^{kt}$$

algebra concept

$e^a \cdot e^c = e^{a+c}$

$$y = \pm e^{C}e^{kt}$$

to get rid of absolute value have ±

$$\boxed{y = Ae^{kt}}$$

looks familiar

constant

Change $\pm e^c$ to A.

$$A(t) = A_0 e^{kt}$$

Amount at time t

original amount

rate

time

Law of Exponential Change: If y changes at a rate proportional to the amount present (that is, if $\boxed{\frac{dy}{dt} = ky}$), and if $y = y_0$ when $t = 0$, then $\boxed{\boldsymbol{y = y_0 e^{kt}}}$. The constant k is the **growth constant** if $k>0$ or the **decay constant** if $k < 0$.

Many of the equations you use in this section will seem familiar and easy. However, each one must be approached from a calculus point of view. Memorize exponential growth formula, the half-life formula and Newton's law of cooling, and be able to apply the ideas covered to new types of problems. For instance, if you're given the differential equation that represents a new type of problem, be able to solve it and answer questions with appropriate interpretations.

Another equation used frequently is compounded interest.

$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ and interest compounded continuously

$A(t) = A_0 e^{rt}$ ← time, rate

original Amount

$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ ← time

rate

principal

100% of principal

number of times compounded

Continously compounded

$A(t) = A_0 e^{kt}$

Example 2

$P = \$800 = A_0$ $r = 6.3\%$ $t = 8\text{yrs}$

$n =$ quarterly

$n = 4$

continously

$A(t) = A_0 e^{kt}$

$800 e^{.063(8)}$

$\boxed{\$1{,}324.26}$

compounded quarterly

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

$800\left(1 + \frac{.063}{4}\right)^{4 \cdot 8}$

$\boxed{\$1319.07}$

Radioactivity

Example 3

$$y = y_0 e^{-kt} \qquad \textit{Solve for } t \textit{ if } y = \frac{1}{2} y_0$$

$$\frac{\frac{1}{2}\cancel{y_0}}{\cancel{y_0}} = \frac{1\cancel{y_0} e^{-kt}}{\cancel{y_0}}$$

$$\frac{1}{2} = e^{-kt}$$

$$\ln\left(\tfrac{1}{2}\right) = \ln e^{-kt}$$

$$\ln\left(\tfrac{1}{2}\right) = (-kt)\ln e \qquad \overset{\ln e}{\log_e e} = 1$$

$$\ln\left(\tfrac{1}{2}\right) = -kt$$

SOLVE for how long is half-life.

$$\frac{\ln\left(\frac{1}{2}\right)}{-k} = t$$

$$\frac{-\ln(2^{-1})}{k}$$

$$\boxed{\frac{\ln 2}{k} = t} \qquad \tfrac{1}{2}\text{ life formula}$$

Example 4

$P_0 = 10 \quad P(30) = 50$ when reach 100

$A(t) = A_0 e^{kt}$

$50 = 10e^{k \cdot 30}$

$5 = e^{30k}$

$\ln 5 = \ln e^{30k}$

$\ln 5 = 30k$

$\frac{\ln 5}{30} = k$

$A(t) = A_0 e^{kt}$ solve for t

$100 = 10e^{kt}$

$10 = e^{kt}$

$\ln 10 = \cancel{\ln e}^{kt}$

$\ln 10 = kt$

$\frac{\ln 10}{k} = t$

$t = \ln 10 \div (\ln 5 \div 30)$

$\boxed{t \approx 42.920 \text{ days}}$

Example 5

carbon 14 $\frac{1}{2}$ life = 5700 years

part 1 Find k

$A(t) = A_0 e^{kt}$

$\frac{1}{2} = e^{k(5700)}$

$\ln(\frac{1}{2}) = \cancel{\ln e}^{5700k}$

$\ln(\frac{1}{2}) = 5700k$

$\frac{\ln(\frac{1}{2})}{5700} = k$

$\frac{\ln(.5)}{5700} = k$

10% has decayed so what's left

$A(t) = 90\%$ of orig.

$90 = 100$ if I had 100 then I'd have 90

$A(t) = A_0 e^{kt}$

$.9 = e^{kt}$

$\ln(.9) = \ln e^{kt}$

$\ln(.9) = kt$

$\frac{\ln(.9)}{k} = t$

$t = \ln(.9) \div (\ln(.5) \div 5700)$

$\boxed{t \approx 866.4176 \text{ yrs}}$

Newton's Law of Cooling:
If T is the Temperature of the object at time t, and T_s is the surrounding temperature, then

$\frac{dT}{dt} = -k(T - T_s)$ and after separating and antidifferentiating the equation the solution is

$\boxed{T - T_s = (T_0 - T_s)e^{-kt}}$ (This is Newton's law of cooling.)

$T(t) = T_s + (T_0 - T_s)e^{-kt}$

Example 6

egg 98°C 18°C water after 5min egg temp is 38°C

How much longer to reach 20°C

$T_s = 18°C \quad T_0 = 98°C$

$$T(t) - T_s = (T_0 - T_s)e^{-kt}$$
$$38 - 18 = (98 - 18)e^{-k(5)}$$
$$20 = 80e^{-5k}$$
$$\frac{20}{80} = e^{-5k}$$
$$\ln(.25) = -5k$$
$$\frac{\ln(.25)}{-5} = k$$

Solve for t when T(t)=20

$$20° - 18° = (98° - 18°)e^{-k\boxed{t}}$$
$$2 = 80e^{-kt}$$
$$\frac{2}{80} = e^{-kt}$$
$$\ln\left(\tfrac{1}{40}\right) = -kt$$
$$\frac{\ln\left(\tfrac{1}{40}\right)}{-k} = t$$

$t = \ln(1 \div 40) \div (-\ln(.25) \div -5)$

13.305 min wrong

13.305 − 5 because it asks how much longer.

≈ 8 more minutes