7.3 calculus notes

Antidifferentiation (Integration) by Parts

One of the most useful derivative formulas is the product rule,

which is $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$

If we integrate with respect to "x" *we* get

$$\int\left(\frac{d}{\cancel{dx}}uv\right)\cancel{dx} = \int\left(u\frac{dv}{\cancel{dx}}\right)\cancel{dx} + \int\left(v\frac{du}{\cancel{dx}}\right)\cancel{dx}$$

Rearranging we get $\int u\frac{dv}{dx}dx = \int\frac{d}{dx}uv\,dx - \int v\frac{du}{dx}dx$

$$\int d\,uv = \int u\,dv + \int v\,du$$

$$uv - \int v\,du = \int u\,dv$$

$$\int u\,dv = uv - \int v\,du$$

If we think of all the d's, dv's and dx's as differentials then we get (from the FTC: differentiation and integration are inverses of one another)

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$$\boxed{\int u\,dv = uv - \int v\,du}$$

"u" dive into ultra violet voodoo

This is the integration by parts formula (Ibp). It expresses one integral in terms of another one. If we choose **u** and **dv** carefully, the second integral may be easier to integrate.

Notice the formula uses **u, v,** and **du**, but the original integral only has **u** and **dv**. So to use the formula we will need to differentiate **u** and antidifferentiate **dv**. This will help us carefully pick **u** and **dv**.

Let's try this on a problem.

Example: $\int \overset{u}{x}\, \underbrace{\overset{dv}{\cos x\, dx}}$

Let **u = x** $du = 1\,dx$ and **dv = cos(x)dx**

This is an easy derivative (**du = dx**) easy antiderivative **v = sin x**

$\frac{dv}{dx} = \cos(x)$

$V = \sin x$

Formula:

$$\int u\, dv = uv - \int v\, du$$

$$\int x \cos dx = x \sin x - \int \sin x\, dx$$

$- \quad -\cos x$

$$\int x \cos x\, dx = x \sin x + \cos x + C$$

This isn't so bad, but what if we chose the wrong **u** and **dv**?

Exploration 1

① $u = 1$ $dv = x\cos x\,dx$
$du = 0$ $\frac{dv}{dx} = x\cos x$
so $v = ?$

② $u = x\cos x$ $dv = 1\,dx$
$du = -x\sin x + \cos x\,dx$ $\frac{dv}{dx} = 1$
$v = x$

$\underset{u}{(x\cos x)}\underset{v}{(x)} - \int v\,du$

$x^2\cos x - \int x(-x\sin x + \cos x)\,dx$

③ $u = \cos x$ $dv = x\,dx$
$du = -\sin x\,dx$ $\frac{dv}{dx} = x$
$v = \frac{x^2}{2}$

④ $u = x$ $dv = \cos x$
Example 1
:)

$\int u\,dv = uv - \int v\,du$

$\frac{x^2}{2}\cos x - \int \frac{x^2}{2} \cdot -\sin x\,dx$

LIPET: If you are wondering what to choose for u, here is what we usually do. Our first choice is a natural logarithm (**L**), if there is one. If there isn't, we look for an inverse trig function (**I**). If there isn't one of these either, look for a polynomial (**P**). Still nothing, look for an exponential (**E**) or a trig function (**T**). That's the preference order: **L I P E T** In general, we want **u** to be something that simplifies when differentiated, and **dv** to be something that remains manageable when integrated.

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$$\int x^2 \cos x\, dx \qquad u = x^2 \qquad dv = \cos x\, dx$$

$$du = 2x\, dx \qquad v = \sin x$$

$$\int x^2 \cos x\, dx = \overset{u}{x^2}\, \overset{v}{\sin x} - \int (\overset{v}{\sin x})\, \overset{du}{2x\, dx}$$

$$= x^2 \sin x - \int 2x \sin x\, dx$$ rewrite

$$= x^2 \sin x - 2\int x \sin x\, dx \qquad u = x \text{ and } dv = \sin x\, dx$$

$$du = dx \text{ and } v = -\cos x$$

$$= x^2 \sin x - 2[\overset{u}{x}\,\overset{v}{(-\cos x)} \overset{-}{-} \overset{\int}{\int} \overset{v}{-\cos x}\, \overset{du}{dx}]$$

$$= x^2 \sin x + 2x \cos x - 2\int \cos x\, dx$$ rewrite distributed

$$x^2 \sin x + 2x \cos x - 2 \sin x + C$$ integrated added C. no more ∫.

On some problems, it seems it doesn't matter what you pick.

#17 $\int e^x \sin x\,dx$

Let u = sin x so du = cos x dx and let dv = e^x dx so v = e^x

$= e^x \sin x - \int e^x \cos x\,dx$

Let u = cos x so du = -sin x dx and let dv = e^x dx so v = e^x

$= e^x \sin x - [e^x \cos x - \int e^x(-\sin x\,dx)]$

$\int e^x \sin x\,dx = e^x \sin x - [e^x \cos x + \int e^x \sin x\,dx]$

$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$

*same *same

$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$

$$\int e^x \sin x\,dx = \frac{e^x \sin x - e^x \cos x}{2} + C$$

In this problem, because the derivatives and integrals of e^x and sin (x) keep reappearing, the original integral reappears and we can solve for it.

If we have integrals where the function we choose for u can be differentiated repeatedly until it becomes zero (i.e. power function) and dv can be easily integrated repeatedly, then we can do Tabular Integration.

This is a fancy way to say integrating from a table. We will now use **u = f** and **dv = g**

On #5 we did $\int x^2 \cos x\, dx$ and used Integration by parts. Now let's use tabular. $u = x^2$ $dv = \cos x\, dx$

f(x) and its derivatives		**g(x) and its integrals**	
x^2	+	cos x	$\int \cos x\, dx$
2x	−	sin x	$\int \sin x\, dx$
2	+	-cos x	$\int -\cos x\, dx$
0	−	-sin x	

Once you have the table the integral is each version of f times each version of g that is one step lower with the attached sign + or - (multiply diagonally). Don't forget "Plus C"

$$\int x^2 \sin x\, dx = x^2 \sin x + 2x\cos x - 2\sin x + C$$

$$x^2 \sin x - 2x(-\cos x) + 2(-\sin x) + C$$

$$x^2 \sin x + 2x\cos x - 2\sin x + C$$

#23 $\int x^3 e^{-2x}\,dx$ $\quad u = x^3 \quad dv = e^{-2x}\,dx$

derive		$\int$	
x^3	$+$	e^{-2x}	$\int e^{-2x}$
$3x^2$	$-$	$\frac{e^{-2x}}{-2}$	$\Rightarrow -\frac{1}{2}e^{-2x}$
$6x$	$+$	$-\frac{1}{2}\cdot\frac{e^{-2x}}{-2}$	$= \frac{1}{4}e^{-2x}$
6	$-$	$\frac{1}{4}\cdot\frac{e^{-2x}}{-2}$	$= -\frac{1}{8}e^{-2x}$
0		$-\frac{1}{8}\cdot\frac{e^{-2x}}{-2}$	$= \frac{1}{16}e^{-2x}$

$$-\frac{1}{2}x^3e^{-2x} - \frac{1}{4}\cdot 3x^2e^{-2x} - \frac{1}{8}\cdot 6x e^{-2x} - 6\cdot\frac{1}{16}e^{-2x} + C$$

$$-\frac{1}{2}x^3e^{-2x} - \frac{3}{4}x^2e^{-2x} - \frac{6}{8}xe^{-2x} - \frac{6}{16}e^{-2x} + C$$

$$e^{-2x}\left(-\frac{x^3}{2} - \frac{3x^2}{4} - \frac{3x}{4} - \frac{3}{8}\right) + C$$

#10 $\int t^2 \ln t \, dt$

t^2	$\ln t$
$2t$	
t	

$u = \ln t \qquad dv = t^2\,dt$

$du = \frac{1}{t}\,dt \qquad \frac{dv}{dt} = t^2$

$V = \frac{t^3}{3}$

$$\int t^2 \ln t\,dt = \underbrace{\ln t\left(\frac{t^3}{3}\right)}_{uv} \underbrace{- \int \frac{\cancel{t^3}^{\,t^2}}{3} \cdot \frac{1}{\cancel{t}}\,dt}_{-\int v\,du}$$

$$\frac{t^3 \ln(t)}{3} - \frac{1}{3}\int t^2\,dt$$

$$\frac{t^3 \ln(t)}{3} - \frac{1}{3} \cdot \frac{t^3}{3} + C$$

$$\boxed{\frac{1}{3}t^3 \ln t - \frac{1}{9}t^3 + C}$$