7.1 notes calculus

Slope Fields and Euler's Method

We have seen problems where we must differentiate. Example: $f(x) = 3x^2 + 2x - 1$.

$f'(x) = 6x + 2$

We have seen problems where we must integrate $\int_1^4 3x^2 + 2x - 1 \, dx$.

$x^3 + x^2 - x \Big|_1^4 \Rightarrow (4^3 + 4^2 - 4) - (1^3 + 1^2 - 1)$

We have seen problems that seem to involve both ideas. Find y if $y' = 3x^2 + 2x - 1$

$\frac{dy}{dx} = 3x^2 + 2x - 1$

$dy = 3x^2 + 2x - 1 \quad dx$

An equation that involves a derivative is called a **Differential Equation.**

Example: Find y if $\frac{dy}{dx} = 3x^2 + 2x - 1$

An equation that involves a derivative is called a **Differential Equation.**

Example: y if $\frac{dy}{dx} = 3x^2 + 2x - 1$

Using the FTC we know derivatives and integrals are inverses of one another so we integrate both sides. $\int \frac{dy}{dx} = \int 3x^2 + 2x - 1$

What's the problem? There is no dx!

Now we integrate.

$$\int dy = \int \left(3x^2 + 2x - 1\right) dx$$

$\int 1 \, dy = \int 3x^2 + 2x - 1 \quad dx$

$y = x^3 + x^2 - x + C$

Is there anything strange about this? (No limits)

Part one of the fundamental theorem of calculus says that each integral has a function that represents it, call it F, any antiderivative. This type of integral is called an **indefinite integral**.

Examples: **Definite** (get an answer) $\int_0^3 x^2\,dx$

Indefinite: (get an antiderivative) $\int x^2\,dx$

$$\int dy = \int (3x^2 + 2x - 1)\,dx \rightarrow y + \cancel{c_1} = x^3 + x^2 - x + c_2$$

$$\cancel{-c_1} \qquad -c_1$$

or $y = x^3 + x^2 - x + c$

$y = x^3 + x^2 - x + \underbrace{(c_2 - c_1)}_{\text{constant}}$

How can we know what "c" is/was? In order to know what "c" is we need an **initial condition**.

Example: $\frac{dy}{dx} = 3x^2 + 2x - 1$ and $y(1) = 2$

when x is 1, y is 2

Separate the differential equation: $dy = (3x^2 + 2x - 1)\,dx$

Integrate using indefinite integral: $\int dy = \int (3x^2 + 2x - 1)\,dx$

$y = x^3 + x^2 - x + c$

Use the initial condition to find c. $y(1) = 2$ when $x = 1$ $y = 2$ and find that $c = 1$

So $\boxed{y = x^3 + x^2 - x + 1}$ this is a **particular solution**.

$y = x^3 + x^2 - x + c$

$2 = 1^3 + 1^2 - 1 + c$

$2 = 1 + c$

$1 = c$

To sum up:
A differential equation (D.E.) is an equation with a derivative in it.

Example: $\frac{dy}{dx} = 2\sin x$ *or* *example*: $y' = x$

$\int dy = \int 2\sin x \, dx$ $\int dy = \int x \, dx$

$y = -2\cos x + C$ $y = \frac{x^2}{2} + C$

A **separable D.E.** is a D.E. where the x's and y's can be separated. (both previous examples are separable D.E.'s)

A **definite integral** has limits of integration and should be evaluated.

Example: $\int_2^x x^2 dx = \frac{x^3}{3} - \frac{8}{3}$

$\left.\frac{x^3}{3}\right|_2^x$

$\frac{x^3}{3} - \frac{2^3}{3}$

An **indefinite integral** does **not** have limits of integration and is "asking" for the antiderivative of the integrand. (Antiderivatives differ by a constant)

Example: $\int x^2 dx = \frac{x^3}{3} + c$

A solution to a D.E. with the arbitrary constant is called a **general solution.** (has $+C$)

A particular solution to a D.E. is a general solution where the constant has been found using an initial condition. (You must have an initial condition to find c.)

Example: Find the function y = f(x) whose derivative is -sin x and whose graph passes through the point (0, 2).

First step: $\frac{dy}{dx} = -\sin x$ *initial condition* $y(0) = 2$

$\int dy = \int -\sin x \, dx$

$y = \cos x + C$

$2 = \cos(0) + C$

$2 = 1 + C$

$1 = C$

$\boxed{y = \cos x + 1}$

Example 3: initial condition (0,3)

$\frac{dy}{dx} = 2x - \sec^2 x$

$\int dy = \int 2x - \sec^2 x \, dx$

$y = \frac{2x^2}{2} - \tan x + C$

$3 = 0^2 - \tan 0 + C$

$3 = C$

$\boxed{y = x^2 - \tan x + 3}$

PSST

Example 4:

$f'(x) = e^{-x^2}$ $f(7) = 3$

$\frac{df}{dx} = e^{-x^2}$

$\int df = \int e^{-x^2} dx$

$\int_a^x e^{-t^2} dt$

$\boxed{f(x) = \int_7^x e^{-t^2} dt + 3}$

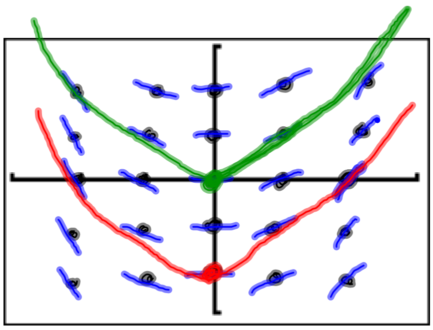
There is also a graphical way of solving D.E.'s, they are called Slope Fields or Direction Fields.

Definition: Slope Field or Direction Field
A slope field for the first order differential equation $\frac{dy}{dx} = f(x, y)$ is a plot of short line segments with slopes f(x, y) for a lattice of points (x, y) in the plane. (See figure 7.2)

This is just a fancy way of saying: take the D.E. and graph it, but only at certain points. This should make intuitive sense because what is a derivative (answer: A slope). Basically, what we would do is make a little slope chart at convenient points on the x-y plane.

Example: $\frac{dy}{dx} = x$ Graph slopes on [-2,2] by [-2,2]

x	y	$\frac{dy}{dx} = x$
-2	-2	-2
-2	-1	-2
(-2	0)	-2
-1	$\mathbb{R}$	-1
0	$\mathbb{R}$	0
1	$\mathbb{R}$	1
2	$\mathbb{R}$	2



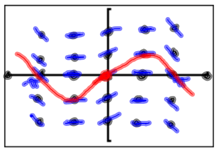
initial condition is (0,-2)
inital condition is (0,0)

$$\int dy = \int x \, dx$$

$$y = \frac{x^2}{2} + C$$

Example 6: Construct a Slope Field for the differential equation $\frac{dy}{dx} = \cos(x)$

x	y	$\frac{dy}{dx} = \cos(x)$
$-\pi$	$\mathbb{R}$	$\cos(-\pi) = -1$
$-\frac{\pi}{2}$	$\mathbb{R}$	$\cos(-\frac{\pi}{2}) = 0$
0	$\mathbb{R}$	$\cos(0) = 1$
$\frac{\pi}{2}$	$\mathbb{R}$	$\cos(\frac{\pi}{2}) = 0$
π	$\mathbb{R}$	$\cos(\pi) = -1$



initial condition (0,0)

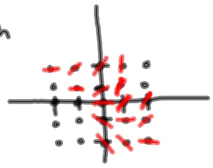
$\int dy = \int \cos(x)\,dx$

$y = \sin x + C$

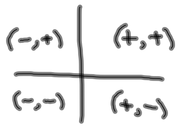
Example 7:

$\frac{dy}{dx} = x + y$ through $(2,0)$

x	y	$x+y$
0	-2	0+-2=-2
0	-1	-1
0	0	0
0	1	1
0	2	2
1	-2	-1
1	-1	0
1	0	1
1	1	2
1	2	3



EXAMPLE 8



Euler's Method

We always need an initial condition when attempting to solve a differential equation, whether analytically, numerically, or graphically. If you think about it, what does a differential equation tell you? (Answer: the slope of the original graph.)

If we have the slope and a starting point then we should be able to string together a bunch of line segments to approximate the original graph. This is kind of what we did with slope fields. Leonhard Euler developed a method for approximating a solution curve. It is called Euler's Method. Here is his idea: It relies heavily on linearizations. $L(x) = f(a) + f'(a)(x - a)$ Read Euler's Method for Graphing a Solution to an Initial Value Problem and look at figures 7.6 and 7.7.

Applying Euler's Method is usually on the AP Exam, so it is important that you understand how to apply this method. You will be given the derivative, the initial condition and dx.

$L(x) = f(a) + f'(a)(x - a)$

$\Delta x = dx$

$$y_{n+1} = y_n + \frac{dy}{dx}(x_n, y_n) \cdot dx$$

$x_0 =$ $y_0 = f(x_0)$

$x_1 = x_0 + dx$ $y_1 =$

$$y_1 = y_0 + \frac{dy}{dx}(x_0, y_0) \cdot dx$$

$x_2 = x_0 + 2dx$ $y_2 =$
$x_1 + dx$

$x_3 = x_0 + 3dx$ $y_3 =$
$x_1 + 2dx$
$x_2 + dx$

$$y_2 = y_1 + \frac{dy}{dx}(x_1, y_1) \cdot dx$$

$$y_3 = y_2 + \frac{dy}{dx}(x_2, y_2) \cdot dx$$

53. $\frac{dy}{dx} = y - x$ and $y = 2$ when $x = 1$. Use Euler's Method with increments of $\Delta x = 0.1$ to approximate the value of y when $x = 1.3$

$f(1) = 2$

$\Delta x = 0.1$

$x_0 = 1$ $y_0 = 2$
$f(1) = 2$

$x_1 = 1.1$ $y_1 = 2.1$

$x_2 = 1.2$ $y_2 = 2.2$

$x_3 = 1.3$ $\boxed{y_3 = 2.3}$

$y_1 = y_0 + (y_0 - x_0)\,dx$

$y_1 = 2 + (2 - 1)(.1)$

$y_1 = 2 + .1$

$y_1 = 2.1$

$y_2 = y_1 + (y_1 - x_1)\,dx$

$y_2 = 2.1 + (2.1 - 1.1)(.1)$

$y_2 = 2.1 + .1$

$y_2 = 2.2$

$y_3 = y_2 + (y_2 - x_2)\,dx$

$y_3 = 2.2 + (2.2 - 1.2)(.1)$

$y_3 = 2.2 + 1(.1)$

$y_3 = 2.2 + .1$

$y_3 = 2.3$