6.5 notes calculus

Trapezoidal Rule

We began this chapter by finding the area bounded by $y = x^2$, the x-axis and the line $x = 3$.

We now know that we can use the Fundamental Theorem of Calculus to find this area. It is $\int_0^3 x^2\,dx$ $\quad \left.\frac{x^3}{3}\right|_0^3 = \frac{3^3}{3} - \frac{0^3}{3}$ $\boxed{9}$

When we first started we used Riemann Sums (the sum of rectangles) to approximate the area.

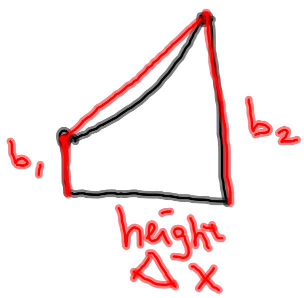
Rectangles don't fit most graphs very well.

Is there a geometric shape that approximates this area better?

In other words, does a shape other than a rectangle work better?

Yes there is. It is a trapezoid.

The area of a trapezoid is: $\boxed{A = \frac{1}{2}(b_1 + b_2)h}$

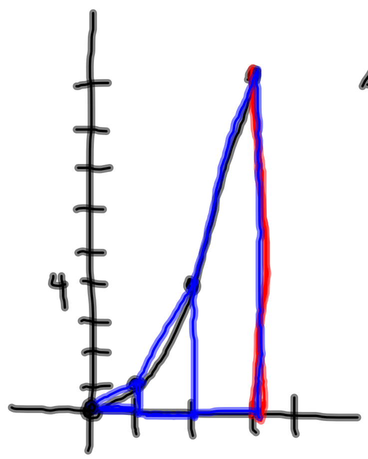


$A_{trap} = \frac{1}{2}h(b_1 + b_2)$

Let's approximate $\int_0^3 x^2\,dx$ using trapezoids and 3 equal subintervals. Draw the graph. Draw the trapezoids. Find the areas of each of the trapezoids. Is the answer we got an over estimate or underestimate?

In LRAM, RRAM, MRAM and Trap Rules use integrand.

when using FTC part II you Antidifferentiate



x	x^2
0	0
1	1
2	4
3	9

$n=3$

$\Delta x\ \dfrac{3-0}{3} \Rightarrow \Delta x = 1$

$[0,1]$ $[1,2]$ $[2,3]$

$\frac{1}{2}(1)[0+1] + \frac{1}{2}(1)[1+4] + \frac{1}{2}(1)(4+9)$

$\frac{1}{2}h(b_1+b_2)$ $\frac{1}{2}h(b_1+b_2)$ $\frac{1}{2}h(b_1+b_2)$

$\frac{1}{2} + \frac{5}{2} + \frac{13}{2} = \frac{19}{2}$

9.5 or $9\frac{1}{2}$

Suppose we want to approximate $\int_a^b f(x)\,dx$ using trapezoids. To find the height of our trapezoids we use $h = \frac{b-a}{n}$ b = end of the interval, a is the start of the interval and n is the number of subintervals.

The trapezoidal Rule: To approximate $\int_a^b f(x)\,dx$, use $\boxed{T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots 2y_{n-1} + y_n\right)}$ where [a,b] is partitioned into n subintervals of equal length $h = \frac{b-a}{n}$. This is also the average of LRAM and RRAM for the same partition. $T = \frac{LRAM_n + RRAM_n}{2}$

Why is this useful? Generally speaking this formula gets us a more accurate approximation in a shorter amount of time. It is also useful when working with diagrams and tables of data where endpoints of subintervals and their values are given.

Example 1

$\int_1^2 x^2\,dx \quad n=4 \quad \frac{2-1}{4}=\frac{1}{4} \quad h=\frac{1}{4}$

$y_1 = x^2$

		x^2
	1	1
$\frac{5}{4}$	$1\frac{1}{4}$	$\frac{25}{16}$
$\frac{3}{2}$	$1\frac{1}{2}$	$\frac{9}{4}=\frac{36}{16}$
$\frac{7}{4}$	$1\frac{3}{4}$	$\frac{49}{16}$
	2	4

$\frac{1}{2}\cdot h\left[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right]$

$\frac{1}{2}\left(\frac{1}{4}\right)\left[1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right]$

$\frac{1}{8}\left(\frac{150}{8}\right) = \frac{150}{64} \Rightarrow \frac{75}{32} \approx 2.34375$

$\left.\frac{x^3}{3}\right|_1^2 \quad \frac{8}{3} - \frac{1}{3} \Rightarrow \frac{7}{3} \quad 2.3\overline{3}$

Example 2

Table

$av(f) \quad \frac{1}{b-a}\int_a^b f(x)\,dx$

12hrs $n=12$

$\frac{12}{12}=1 \quad h=1$

$\frac{1}{12}\left[\frac{1}{2}(1)\left[63 + 2(65) + 2(66) + 2(68) + 2(70) + 2(69) + 2(68) + 2(68) + 2(65) + 2(64) + 2(62) + 2(58) + 55\right]\right]$

$\frac{1}{12}\left(\frac{1}{2}\left[1564\right]\right)$

$\frac{1}{12}(782) \approx 65.17$

average

There is one more way to approximate the area under a curve. Since functions are usually curves, it might make sense to use a curve to approximate the area. What is the simplest curve we use? parabola

Formula for the Area under a parabolic arc is $A_P = \frac{h}{3}(l + 4m + r)$ where h is half the length of the base, l and r are the lengths of the left and right sides, and m is the altitude at the midpoint of the base.

Simpson's Rule:

To approximate $\int_a^b f(x)\,dx$,

use $S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$,

$\frac{1}{3}\left(\frac{b-a}{n}\right)($

where [a, b] is partitioned into an even number n of subintervals of equal length $h = \frac{b-a}{n}$

Example 3

$\int_0^2 5x^4 dx \quad [0,2] \quad n=4 \quad \frac{2-0}{4}=\frac{1}{2} \quad h=\frac{1}{2}$

$y_1 = 5x^4$

	$5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}=1\frac{1}{2}$	$\frac{405}{16}$
2	80

$S = \frac{1}{3}h\left[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right]$

$\frac{1}{3}\left(\frac{1}{2}\right)\left[0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right]$

$\frac{1}{6}\left(\frac{5}{4} + \frac{40}{4} + \frac{405}{4} + \frac{320}{4}\right) \Rightarrow \frac{385}{12}$

≈ 32.083

$\int_0^2 5x^4 dx$

$x^5\Big|_0^2 \quad 32 - 0 = 32$

Now let's read "Error Analysis"

Problem #19

$\int_{-1}^{3} x^3 - 2x \; dx \quad n=4 \quad h=1$

$\frac{b-a}{n} \quad \frac{3--1}{4} = \frac{4}{4} = 1$

x	x^3-2x	
-1	1	y_0
0	0	y_1
1	-1	y_2
2	4	y_3
3	21	y_4

a) $S = \frac{1}{3}(1)\left[1 + 4(0) + 2(-1) + 4(4) + 21\right]$

$\frac{1}{3}\left[1 + 0 - 2 + 16 + 21\right]$

$\frac{1}{3}(36)$

$= 12$

b) $\int_{-1}^{3} x^3 - 2x \, dx$

$\frac{x^4}{4} - \frac{\cancel{2}x^2}{\cancel{2}}\Big|_{-1}^{3}$

$\left(\frac{3^4}{4} - 3^2\right) - \left(\frac{(-1)^4}{4} - (-1)^2\right)$

$\left(\frac{81}{4} - 9\right) - \left(\frac{1}{4} - 1\right)$

$\frac{81}{4} - 9 - \frac{1}{4} + 1$

$\frac{80}{4} - 8 = \boxed{12}$

c) Error Bound

$|E_S| \leq \frac{b-a}{180} h^4 M_{f^4} \longleftarrow$ max value of f^4 on $[a,b]$

h: length of each subinterval

$f(x) = x^3 - 2x$ integrand

$f'(x) = 3x^2 - 2$

$f''(x) = 6x$

$f'''(x) = 6$

$f^4(x) = 0$

$|E_S| \leq \frac{4}{180}(1)^4(0)$

$|E_S| \leq 0$

d) yes $f^4(x)$ of a cubic is always "0".
Simpson's rule will be exact.