6.4 notes calculus

Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is the theorem (idea) that connects the two branches of calculus—**differential calculus** (derivatives, differentials, rates of change, how fast we are going) and **integral calculus** (areas and accumulation, how far have we gone)

★ on the TEST

The Theorem comes in two parts:

Part I:

If f is continuous on [a,b], then the function $F(x)=\int_a^x f(t)\,dt$ has a derivative at every point in the interval [a,b], namely f(x) F′(x) = f(x) {x is the upper limit of integration} and

$$F'(x)=\frac{dF}{dx}=\frac{d}{dx}\int_a^x f(t)\,dt=f(x)$$

This theorem has unbelievable power! It tells us that the derivative of the integral is a function of the upper limit. The process of integration and differentiation are inverses!

Part I if $F(x)=\int_a^x f(t)\,dt$

then $F'(x)=\frac{d}{dx}\int_a^x f(t)\,dt=f(x)$

How do we apply this?

Example: Find

$$\frac{d}{dx}\int_\pi^x \tan t\ dt$$

$F'(x)=\tan x$

answer is: $f=\tan x$

$$\frac{d}{dx}\int_2^x (t^2+t)\ dt$$

$\int_2^x t^2+t\ dt$

$\left.\frac{t^3}{3}+\frac{t^2}{2}\right|_2^x$

$\frac{d}{dx}\left[\left(\frac{x^3}{3}+\frac{x^2}{2}\right)-\left(\frac{2^3}{3}+\frac{2^2}{2}\right)\right]$

$\frac{1}{3}x^3$ $\frac{1}{2}x^2$ constant

$\cancel{3}\cdot\frac{1}{\cancel{3}}\cdot x^{3-1}$ $2\cdot\frac{1}{2}x^{2-1}$ -0

x^2+x

answer is: $\boxed{f=x^2+x}$

What do you need to understand about the first part? The derivative of an integral is the integral function with the upper limit plugged in.

Examples:

1. $\dfrac{d}{dx}\displaystyle\int_0^x \sin t \; dt$ $\quad \sin x$

2. $\dfrac{d}{dx}\displaystyle\int_3^x \frac{1}{t+2} \; dt$ $\quad \dfrac{1}{x+2}$

What if the upper limit itself is a function?
We will need to use the chain rule.

$F'(g)\cdot g'$ so $\dfrac{d}{dx}\displaystyle\int_1^{x^2} \sin t \; dt = \sin x^2 \cdot 2x = 2x\sin x^2$

$\dfrac{d}{dx}\displaystyle\int_1^{x^2} \sin t \, dt \;\; \dfrac{d}{dx}\left(-\cos t\Big|_1^{x^2}\right) \Rightarrow \dfrac{d}{dx}\left(-\cos(x^2) - -\cos(1)\right)$ $\quad \sin(x^2)\cdot 2x$

In general $\boxed{\dfrac{d}{dx}\displaystyle\int_c^u f(t) \; dt = f(u)\cdot\frac{du}{dx}}$

Practice problems

Find $\dfrac{dy}{dx}$

11. $y = \displaystyle\int_2^{5x} \frac{\sqrt{1+u^2}}{u} \, du$

$\dfrac{dy}{dx} = \dfrac{d}{dx}\displaystyle\int_2^{5x} \frac{\sqrt{1+u^2}}{u} \, du$

$\dfrac{\sqrt{1+(5x)^2}}{5x} \cdot \dfrac{d}{dx}(5x)$

$\dfrac{\sqrt{1+25x^2}}{\cancel{5}x} \cdot \cancel{5} \Rightarrow \boxed{\dfrac{\sqrt{1+25x^2}}{x}}$

9. $y = \displaystyle\int_0^{x^2} e^{t^2} \; dt$

$\dfrac{dy}{dx} = \dfrac{d}{dx}\displaystyle\int_0^{x^2} e^{t^2} \, dt$

$e^{(x^2)^2} \cdot \dfrac{d}{dx}(x^2)$

$e^{x^4} \cdot 2x$

$\boxed{2xe^{x^4}}$

What if the lower limit is x?

Look at Example 3

$$\frac{d}{dx}\int_x^5 3t\sin t\,dt$$

$$-\frac{d}{dx}\int_5^x 3t\sin t\,dt \Rightarrow \boxed{-3x\sin x}$$

13. $\frac{d}{dx}\int_x^6 \ln(1+t^2)\,dt$

$$-\frac{d}{dx}\int_6^x \ln(1+t^2)\,dt \Rightarrow \boxed{-\ln(1+x^2)}$$

15. $y=\int_{x^3}^5 \frac{\cos t}{t^2+2}\,dt$

$$\frac{dy}{dx}=\frac{d}{dx}\int_{x^3}^5 \frac{\cos t}{t^2+2}\,dt$$

$$-\frac{d}{dx}\int_5^{x^3} \frac{\cos t}{t^2+2}\,dt$$

$$-\frac{\cos(x^3)}{(x^3)^2+2}\cdot\frac{d}{dx}(x^3)$$

$$-\frac{\cos(x^3)}{x^6+2}\cdot 3x^2$$

$$\boxed{\frac{-3x^2\cos(x^3)}{x^6+2}}$$

What if both limits are functions of x?

$$\frac{d}{dx}\int_x^{x^2}(3t^2-2t)\,dt = \frac{d}{dx}\left(\int_0^{x^2}(3t^2-2t)\,dt-\int_0^x(3t^2-2t)\,dt\right)$$

$$=\left(3(x^2)^2-2(x^2)\right)2x-(3x^2-2x)$$

$$=(3x^4-2x^2)(2x)-3x^2+2x$$

$$=6x^5-4x^3-3x^2+2x$$

$$\frac{d}{dx}\int_x^{x^2}(3t^2-2t)\,dt$$

Additivity

$$\frac{d}{dx}\left[\int_x^0(3t^2-2t)\,dt+\int_0^{x^2}(3t^2-2t)\,dt\right]$$

$$\frac{d}{dx}\left[-\int_0^x(3t^2-2t)\,dt+\int_0^{x^2}(3t^2-2t)\,dt\right]$$

$$-(3x^2-2x)+(3(x^2)^2-2x^2)\cdot\frac{d}{dx}(x^2)$$

$$-3x^2+2x+(3x^4-2x^2)(2x)$$

$$-3x^2+2x+6x^5-4x^3$$

$$\boxed{6x^5-4x^3-3x^2+2x}$$

Ex 3b $\frac{d}{dx}\int_{2x}^{x^2}\frac{1}{2+e^t}\,dt$

$$\frac{d}{dx}\left[\int_{2x}^0\frac{1}{2+e^t}\,dt+\int_0^{x^2}\frac{1}{2+e^t}\,dt\right]$$

$$\frac{d}{dx}-\int_0^{2x}\frac{1}{2+e^t}\,dt+\frac{1}{2+e^{x^2}}\cdot 2x$$

$$-\frac{1}{2+e^{2x}}\cdot 2+\frac{2x}{2+e^{x^2}}$$

$$\boxed{\frac{-2}{2+e^{2x}}+\frac{2x}{2+e^{x^2}}}$$

Example 4

Chapter 7 will give us some idea of how to come up with the functions that produce this derivative. The correct function and its values can be easily computed with a calculator or computer.

$y = f(x)$

$\frac{dy}{dx} = \tan x$ $\quad f(3) = 5$ when $x = 3$, $y = 5$

$\frac{d}{dx}\int_a^x \tan t \, dt = \tan x$

$\frac{d}{dx}\int_a^3 \tan t \, dt =$

when $x=3$ $y=5$

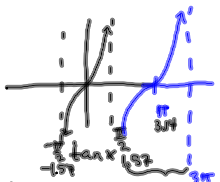
$\int_3^3 \tan t \, dt = 0$

$\int_3^3 \tan t \, dt + 5 = 5$

$f(x) = \int_3^x \tan t \, dt + 5$

Read and try: **Graphing the Function** $\int_a^x f(t)\,dt$

Exploration 1



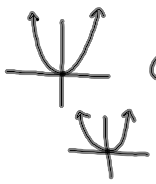
V.A. $x = \frac{\pi}{2} + \pi k$

window:

Exploration 2

① $y_1 = \text{math } 8\,(\text{math } 9,\ x^2, x, 0, x), x, x$

$\frac{d}{dx}\left(\int_0^x x^2\,dx\right)\Big|_{\boxed{x}=x}$



② $y_1 = \frac{d}{dx}\left(\int_5^x x^2\,dx\right)\Big|_{x=x}$

change 0 to 5

③ $\int_0^x x^2\,dx$ xint $= 0$

④ $\int_5^x x^2\,dx$ xint $= 5$

⑤ $y = \frac{d}{dx}\int_a^x f(t)\,dt$

a gives no change

⑥ $y = \int_a^x f(t)\,dt$

a changes the x-int

Fundamental Theorem of Calculus Part 2

The first part of the Fundamental Theorem of Calculus tells us how derivatives and integrals are related, namely, they are inverses of one another. The second part tells us how we can use this to evaluate integrals.

If *f* is continuous at every point of [a,b], and if F is any antiderivative of *f* on [a,b], then

$\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any anti-derivative of *f*

* if $F'(x) = f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$

This part of the Fundamental Theorem is called the Integral Evaluation Theorem.

Let's take a look at the proof.

Example:

$$\int_2^3 x^2\,dx = \frac{x^3}{3} = \left[\frac{3^3}{3} - \frac{2^3}{3}\right]_2^3 = \frac{19}{3}$$

Confirm with fnint

Example:

$$\int_{-3}^{6} \left(2x^3 - 4x + 1\right)dx = \frac{2x^4}{4} - \frac{4x^2}{2} + x = \frac{x^4}{2} - 2x^2 + x$$

$$\left.\frac{x^4}{2} - 2x^2 + x\right|_{-3}^{6}$$

$$\left(\frac{6^4}{2} - 2(6)^2 + 6\right) - \left(\frac{(-3)^4}{2} - 2(-3)^2 + -3\right)$$

$$\boxed{562.5}$$

Example 5

$$\int_{-1}^{3} (x^3+1)\,dx$$

$$\left.\frac{x^4}{4} + x\right|_{-1}^{3}$$

$$\boxed{\left(\frac{3^4}{4} + 3\right) - \left(\frac{(-1)^4}{4} + -1\right)} \Rightarrow \boxed{24}$$

Remember: the integral started out as a way to find the accumulated or net area under a curve. If function values are negative the integral will be negative. But area is always positive. When we say "area" under a curve we mean total area.

Example 6:

Find the area under the curve of $y = 4 - x^2$ from $0 \le x \le 3$

$4x - \frac{x^3}{3}\Big|_0^3 \Rightarrow 12 - 9 = \boxed{3}$

$\int_0^3 4 - x^2\,dx \Rightarrow$ net area not total area!

$\int_0^2 4 - x^2\,dx + -\int_2^3 4 - x^2\,dx$

$4x - \frac{x^3}{3}\Big|_0^2 + -\left(4x - \frac{x^3}{3}\Big|_2^3\right)$

$\left(8 - \frac{8}{3}\right) - \cancel{\left(4(0) - \frac{0^3}{3}\right)} + -\left[\overbrace{\left(4(3) - \frac{3^3}{3}\right)}^{F(b)} - \overbrace{\left(4(2) - \frac{2^3}{3}\right)}^{F(a)}\right]$

$12 - 9$ $\quad -\left(8 - \frac{8}{3}\right)$

$8 - \frac{8}{3} + -\left(3 - 8 + \frac{8}{3}\right)$

-5

$8 - \frac{8}{3} + 5 - \frac{8}{3}$

$\boxed{\frac{23}{3}}$

Example: Find the area under the curve of $y = x^3$ on $[-1, 3]$

$\int_{-1}^3 x^3\,dx$

$-\int_{-1}^0 x^3\,dx + \int_0^3 x^3\,dx$

$-\left(\frac{x^4}{4}\Big|_{-1}^0\right) + \frac{x^4}{4}\Big|_0^3$

$-\left(\cancel{\frac{0^4}{4}} - \frac{(-1)^4}{4}\right) + \left(\frac{3^4}{4} - \cancel{\frac{0^4}{4}}\right)$

$\frac{1}{4} + \frac{81}{4} = \frac{82}{4}$ $\boxed{20.5}$

There are some general tips to help find the area.

How to Find Total Area Analytically

To find the area between the graph of $y = f(x)$ and the x-axis over the interval [a,b] analytically,

1. Partition [a,b] with the zeros of f (Find the x-intercepts of the function – where function crosses x-axis)
2. Integrate f over each subinterval
3. Add the absolute values of the integrals

How the Find Total Area Numerically

To find the area between the graph of $y = f(x)$ and the x-axis over the interval [a,b] numerically, evaluate

fnInt ($|f(x)|$, x, a, b)

$\int_0^3 |4 - x^2|\,dx$

fnInt $(|4 - x^2|, x, 0, 3) = \frac{23}{3}$

Example 7 Total AREA

in calc $\int_{-3}^{3} |x\cos 2x|\, dx$

fnint(|x cos(2x)|, x, −3, 3) $\boxed{5.425}$

Example 8

Graph is $f(x)$

$h(x) = \int_1^x f(t)\,dt$

a) Find $h(1)$

$h(1) = \int_1^1 f(t)\,dt$ $\boxed{0}$

b) is $h(0)$ positive or negative

$\int_1^0 f(t)\,dt$ order of integration

$\int_0^1 f(x)\,dx$ is positive

c) Find value of x where $h(x)$ is a max.

$h'(x) = \frac{d}{dx}\int_1^x f(t)\,dt$

$h'(x) = f(x) \Longleftarrow f(x)$ is graph.

where $h'(x) = 0$ $f(x) = 0$

0 (0,1) 1 (1,4) 4 max (4,8) 8

$h'(x)$ / $f(x)$: + + −

$\boxed{\text{max is at } x = 4}$

d) min happens when $h'(x)$ goes from neg to pos

does not happen at $x = 1$ or $x = 4$

So check endpoints

$\int_0^1$ positive $h(x) = \int_1^8$ is negative

min occurs at $x = 8$ because $\int_1^8 f(t)\,dt$ is more negative than $\int_1^0 f(t)\,dt$

e) $h'(x) = f(x)$

$h''(x) = f'(x)$

0 (0,1) 1 (1,3) 3 (3,6) 6 (6,8) 8

$f'(x)$: − + − +

$0 = f'(x)$

points of inflection occur at $x = 1$, $x = 3$, $x = 6$