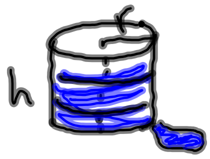
5.6 notes calculus

Related Rates (Related Derivatives)
Equations relating derivatives

Example: A cylindrical tank leaks water at 1 cubic meter/hour. What is the rate at which the height of the water is decreasing?



$\frac{dV}{dt} = \frac{-1\ m^3}{hr}$ $V = \pi r^2 h$

variable is h
h(t)

Find $\frac{dh}{dt}$

$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$

radius is constant

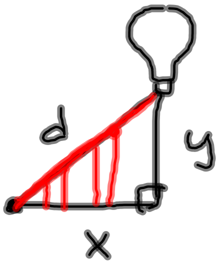
$-1 = \pi r^2 \frac{dh}{dt}$

$\boxed{\frac{-1}{\pi r^2} = \frac{dh}{dt}}$

$\frac{dh}{dt} = \frac{-1}{\pi r^2}$ m/hr

Example: A hot air balloon is rising and being pushed away by the wind. At what rate is the distance changing?

$\frac{dd}{dt}$



$d^2 = x^2 + y^2$

x and y are in terms of t.

$d = \sqrt{x^2+y^2}$

$\frac{dd}{dt} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$\frac{dd}{dt} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2\sqrt{x^2+y^2}}$

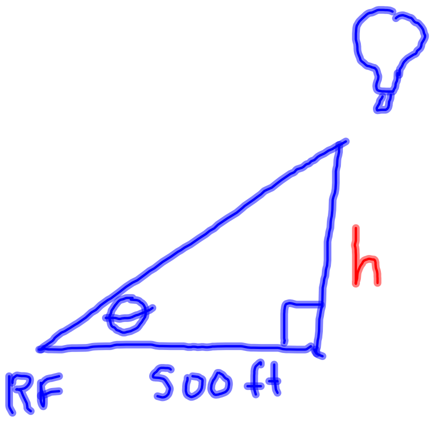
$\boxed{\frac{dd}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}}}$

Related Rate Problem Strategy:

1. ***Draw a picture and name the variables and constants. Use t for time.*** Assume all variables are differentiable functions of t.
2. ***Write down the numerical information*** (in terms of the symbols you have chosen).
3. ***Write down what we are asked to find*** (usually a rate, expressed as a derivative).
4. ***Write an equation that relates the variables.*** You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. ***Differentiate both sides of the equation implicitly with respect to t.*** Then express the rate you want in terms of the rate and variables whose values you know.
6. ***Evaluate and interpret.*** Use known values to find the unknown rate.

Example 2 "A Rising Balloon"

p251

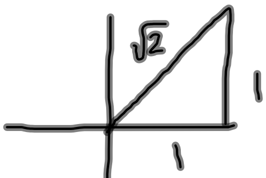


$\theta = \frac{\pi}{4}$ $\quad \frac{d\theta}{dt} = .14 \, \frac{\text{rad}}{\text{min}}$

Find $\frac{dh}{dt}$

$$\tan\theta = \frac{h}{500}$$

$$500\tan\theta = h$$



$$h = 500\tan\theta$$

$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\sec\frac{\pi}{4} = \sqrt{2}$

$$\frac{dh}{dt} = 500\sec^2(\theta)\cdot\frac{d\theta}{dt}$$

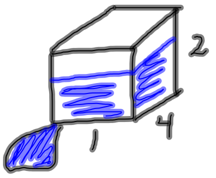
$$\frac{dh}{dt} = 500\cdot\left(\sec\frac{\pi}{4}\right)^2\cdot(.14)$$

$$\frac{dh}{dt} = 500(\sqrt{2})^2(.14)$$

$$1000(.14)$$

$$\boxed{\frac{dh}{dt} = 140 \text{ ft/min}}$$

Example: The base of a rectangular tub has length equal to one meter and width equal to four meters. The height of the tub is two meters. Water is flowing through a leak in the bottom of the tub at a rate of one cubic meter per hour. What is the rate at which the water level is decreasing?



$\frac{dV}{dt} = -1 \frac{m^3}{hr}$ $\frac{dh}{dt} = ?$

$V = Lwh$

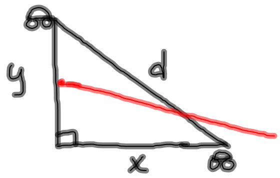
$V = 1(4)h$

$\frac{dV}{dt} = 4\frac{dh}{dt}$

$-1 = 4\frac{dh}{dt}$

$\boxed{-\frac{1}{4}\ m/hr = \frac{dh}{dt}}$

Example 3: "A Highway Chase" p252



when y is .6 miles x is .8 miles $\frac{dd}{dt} = 20mph$

cruiser $-60mph = \frac{dy}{dt}$ Find $\frac{dx}{dt}$

$d^2 = x^2 + y^2$

$d = \sqrt{x^2+y^2}$

$\frac{dd}{dt} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2\sqrt{x^2+y^2}}$

$\frac{dd}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}}$

$20 = \frac{.8\frac{dx}{dt} + .6(-60)}{\sqrt{.8^2+.6^2}}$

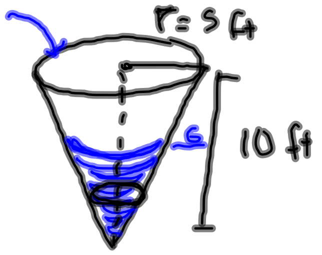
$20 = .8\frac{dx}{dt} + .6(-60)$

$20 + 36 = .8\frac{dx}{dt}$

$\frac{56}{.8} = \frac{dx}{dt}$

$\boxed{70\ mph}$

Example 4: Filling a conical tank



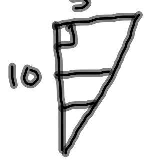
$$\frac{dV}{dt} = 9\,\frac{ft^3}{min}$$

$h=6$

Find $\frac{dh}{dt}$

relationship of the h and r

$r=5 \quad h=10$



$$\frac{5}{10} = \frac{r}{h}$$

$$\frac{1}{2}h = r$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{1}{2}h\right)^2 \cdot h$$

$$V = \frac{1}{3}\pi \left(\frac{1}{4}h^2\right) \cdot h$$

$$V = \frac{\pi}{12} h^3$$

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \cdot \frac{dh}{dt}$$

$$9 \cdot \frac{12}{\pi} \cdot \frac{1}{3} \cdot \frac{1}{h^2} = \frac{dh}{dt}$$

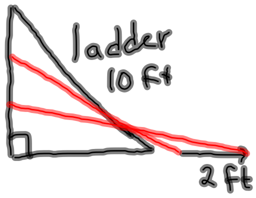
$$\frac{36}{\pi h^2} = \frac{dh}{dt}$$

$$\frac{36}{\pi (6)^2} = \frac{dh}{dt}$$

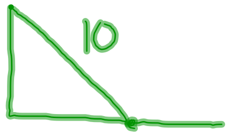
$$\frac{1}{\pi}\ ft/min$$

$$\boxed{\approx .318\ ft/min}$$

Exploration #1 Sliding Ladder



$x_1(t) = 2T$
$y_1(t) = 0$
} "horizontal place of ladder ground"



$x_2(t) = 0$
$y_2(t) = \sqrt{10^2 - (2T)^2}$

height the ladder

$a^2 + b^2 = c^2$
$(2T)^2 + b^2 = 10^2$
$h^2 = 10^2 - (2T)^2$
$h = \sqrt{10^2 - (2T)^2}$

min $T = 0$
max $T = 5$

$y = \sqrt{100 - 4T^2}$

$$\frac{dy}{dt} = \frac{1 \cdot -8T}{2\sqrt{100-4T^2}} \qquad \frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \frac{-4T}{\sqrt{100-4T^2}} \qquad \left.\frac{dy}{dt}\right|_{t=3} \frac{-12}{\sqrt{100-36}} \quad \frac{-12}{8}$$

$-\frac{3}{2}$ ft/sec²

$$\lim_{t \to 5^-} \frac{dy}{dt} = -\infty$$

at $t = 5$ $\quad \dfrac{-4(5)}{\sqrt{100-4(25)}} \quad \dfrac{-25}{0} \quad ?$

at $t = 4.99$

$$\frac{-4(4.99)}{\sqrt{100-4(4.99)^2}} \approx -31.575 \text{ ft/sec}^2$$

$$\frac{-4(4.999)}{\sqrt{100-4(4.999)^2}} \approx -99.984 \text{ ft/sec}^2$$