5.5 notes calculus

Linearization and Differentials

We have zoomed in on many graphs for many reasons. We always find that they start to look like lines. In calculus we know they start to look like the tangent line at that point. This leads to an important idea.

Find the equation of the tangent line for $y = x^2$ at $x = \frac{1}{3}$

The point to use would be $(\frac{1}{3}, \frac{1}{9})$

Use point-slope form: $y = m(x - x_1) + y_1$ or $y = y_1 + m(x - x_1)$

$y - y_1 = m(x - x_1)$

Say we have $y = f(x)$ at $x = a$ The point would be $(a, f(a))$ and the slope (m) would be $f'(a)$.

Using point slope form we have $\boxed{y = f(a) + f'(a)(x - a)}$.

You are going to see this formula a lot!

$(a, f(a))$

$$y = y_1 + m(x - x_1)$$
$$= f(a) + f'(a)(x - a)$$

We use this idea as a **linear approximation** for the function **close to the point of tangency**.

If f is differentiable at $x = a$, then the approximating function $L(x) = f(a) + f'(a)(x - a)$ is the **linearization** of f at $x = a$.

$L(x)$ is the **standard linear approximation**.

The **center of approximation** is "a" or $x = a$.

The closer we are to "a" the better the approximation.

Find a linearization for $f(x)$ and calculate its accuracy for $\Delta x = .01$

$f(x) = x^3 - 2x + 3 \qquad a = 2$ $\qquad (2,7) \qquad f(2)=7$

$f'(x) = 3x^2 - 2$

$f(2) = 8 - 4 + 3 = 7 \qquad f'(2) = 3(4) - 2 = 10$ $\qquad f'(2) = 10$

Use formula $L(x) = f(a) + f'(a)\cdot(x-a)$

$L(x) = 7 + 10(x-2)$

$= 7 + 10x - 20$

$= 10x - 13$ $\qquad L(x) = 10x - 13$

Look at the function and how accurate the approximation 10x-13 is for values of x near 2. (2.01)

$f(2.01) = (2.01)^3 - 2(2.01) + 3$

$f(2.01) = 7.100601$

$L(2.01) = 10(2.01) - 13$

7.1

#2 page 246 $\quad f(x) = \sqrt{x^2+9} \qquad a = -4$

$f(-4) \Rightarrow f(-4) = \sqrt{(-4)^2+9}$

$f(-4) = 5$

$f'(x) = \frac{1}{2\sqrt{x^2+9}} \cdot 2x \qquad f'(x) = \frac{x}{\sqrt{x^2+9}}$

$f'(-4) = \frac{-4}{\sqrt{(-4)^2+9}}$

$f'(-4) = \frac{-4}{5}$

$L(x) = 5 + \frac{-4}{5}(x+4)$

$L(x) = -\frac{4}{5}x + -\frac{16}{5} + \frac{25}{5}$

$L(x) = -\frac{4}{5}x + \frac{9}{5}$

$L(a+.1) \qquad L(-3.9) = -\frac{4}{5}(-3.9) + \frac{9}{5} \quad \boxed{4.92}$

$L(-4+.1) \qquad f(-3.9) \approx 4.92036584$

Example #2 (page 238) $f(x) = \cos(x) \qquad a = \pi/2$

$f(x) = \cos(x) \qquad (\frac{\pi}{2}, 0)$

$f(\frac{\pi}{2}) = \cos(\frac{\pi}{2})$

$f'(x) = -\sin(x)$

$f'(\frac{\pi}{2}) = -1$

$L(x) = 0 + -1(x - \frac{\pi}{2})$

$\boxed{L(x) = -x + \frac{\pi}{2}}$

$f(1.75) \approx -.1782460556$

$L(1.75) \approx -.17920367$

Definition: Differentials: Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is $dy = f'(x)\,dx$

Differentials: derivatives are rates $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ we can use change in y and change in x as variables, so dy and dx are called differentials. (Don't forget rules of differentiation!)

Example: $y = x^2$ Answer: $dy = (2x)\,dx$

$\frac{dy}{dx} = 2x$

$dy = 2x\,dx$

Example: $x^2 + y^2 = 2$

$2x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{2y}$

$\boxed{dy = -\frac{x}{y}\,dx}$

Example: $y = x^5 + 37x$

$\frac{dy}{dx} = 5x^4 + 37$

$dy = (5x^4 + 37)\,dx$

Example $y = \sin(3x)$

$\frac{dy}{dx} = 3\cos(3x)$

$dy = 3\cos(3x) \cdot dx$

Example $d(\tan 2x)$

$f(x) = \tan(2x)$

$\frac{df}{dx} = 2\sec^2(2x)$

$\boxed{df = 2\sec^2(2x)\,dx}$

Differentials are variables that represent change.
Sometimes we write $df = f'(x)\,dx$ in place of $dy = f'(x)\,dx$ calling df the differential of f.

Estimating Change with Differentials
Suppose we know the value of a differentiable function $f(x)$ at a point "a" and we want to predict how much this value will change if we move to a nearby point $a + dx$. If dx is small, f and its linearization L at a will change by nearly the same amount. (Figure 5.47 page 241) Since the values of L are simple to calculate, calculating the change in L offers a practical way to estimate the change in f.

Differential Estimate of Change: Let $f(x)$ be differentiable at $x = a$. The approximate change in the value of f when x changes from a to $a + dx$ is $df = f'(a)\,dx$

$df = f(a+dx) - f(a)$ or $dL = f'(a)\,dx$ for small values of dx.

$dL = df$ which means $df = f'(a)\,dx$

Example 7 (page 242)

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

from 10 to 10.1

$r = 10$ $dr = .1$

$dA = 2\pi r\, dr$

$dA = 2\pi(10)(.1)$

$dA = 2\pi\ m^2$ estimated change

find the true change

$A = \pi(10)^2$ $A = \pi(10.1)^2$

$A = 100\pi$ $A = 102.01\pi$

$102.01\pi - 100\pi$

Actual change 2.01π

error is $2.01\pi - 2\pi = \boxed{.01\pi}$

Absolute Change: $\Delta f = f(a+dx) - f(a)$

Estimated Change: $df = f'(a)\,dx$

Approximation Error: $|\Delta f - df|$

#27 page 246

$f(x) = x^2 + 2x$ $a = 0,\quad dx = .1$

$f'(x) = 2x + 2$

a. *absolute change* $\Delta f = f(.1) - f(0) = ((.1)^2 + 2(.1)) - ((0)^2 + 2(0)) = .21 - 0 = .21$

b. $df = \underset{2x+2}{f'(a)}dx$ $dx = .1$

$df = (2(0) + 2)\cdot(.1)$ $a = 0$

$\boxed{df = .2}$ estimated change using differential

c. $|\Delta f - df| = |.21 - .2| = .01$

$\Downarrow$

$|.21 - .2| = .01$

Example 9, 10, 11, and 12 (Real life applications of calculus)

Read them

Newton's Method -- a method for finding roots

Guess a first approximation to a solution of the equation f(x) = 0.

A graph of y = f(x) may help.

Use the first approximation to get a second, the second to get a third, and so on, using the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Newton's Method using your calculator:

Step 1: On home screen Store your guess in x. guess→x

Step 2: In graphing menu y =

Let y_1= function; and y_2 = nderiv (y_1,x, x)

Step 3: x-(y_1/ y_2) →x Press enter key over and over and watch numbers converge to the zero of the function.

Look at Figure 5.49, 5.50. Read through **Example 13**.

#53. $x^3 + x - 1 = 0$

$y_1 = x^3 + x - 1$

home screen guess 1.3 $\quad 1.3 \xrightarrow{sto} x$

$y_2 = \text{nderiv}(y_1, x, x)$ or $y_2 = \frac{d}{dx}(y_1)\Big|_{\boxed{x}=x}$

home screen

$x - (y_1 \div y_2) \xrightarrow{sto} x$ enter

zero .682327

Because the formula is divided by $f'(a)$ if $f'(a) = 0$ it will not work. Graphically what does $f'(a) = 0$ mean?

There are times when Newton's method will skip a root, or not work at all. Look at Figure 5.52, 5.53