5.2 notes calculus

Mean Value Theorem

The average rate of change is something that we have calculated for a long time

$\frac{f(b)-f(a)}{b-a}$ slope of secant line (x_1, y_1) (x_2, y_2) $\frac{y_2-y_1}{x_2-x_1}$

But, what we study in calculus is instantaneous rates of change (derivative). The mean value theorem connects average rate of change to the derivative and it leads to many important applications.

(Rolle's Theorem) figure 5.10 (p200)

Theorem 3: Mean Value Theorem for Derivatives

If y = f(x) is continuous at every point of the closed interval [a,b] and differentiable at every point of its interior (a,b), then there is at least one point c in (a,b) at which $f'(c) = \frac{f(b)-f(a)}{b-a}$

In other words: there is at least one point where the derivative equals the average rate of change.

Now let's look at figure 5.11

Show that the function f satisfies the hypotheses of the MVT on the given interval and Find each value of c in (a,b) that satisfies the equation.

Example: $f(x) = x^2 + 2x + 1$ [1,2]

$f(1) = 4$

$f(2) = 9$

$\frac{9-4}{2-1} = \frac{5}{1}$

$f'(x) = 2x+2$

$2x+2 = 5$

$2x = 3$

$x = \frac{3}{2}$

Look at Example 3

$f(x) = \sqrt{1-x^2}$ $[-1,1]$

$f(-1) = 0$

$f(1) = 0$

$\frac{0-0}{1--1}$ $\frac{0}{2} = 0$

$f'(x) = \frac{1}{2\sqrt{1-x^2}} \cdot -2x$

$\frac{-x}{\sqrt{1-x^2}} = 0$

$-x = 0$

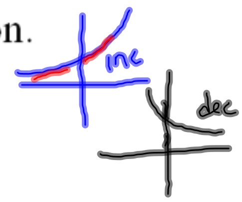
$\boxed{x = 0}$

Look at Example 4

Read it.

Increasing and decreasing functions

Draw an increasing and decreasing function.



As we move from left to right

f is increasing on I $\quad x_1 < x_2 \longrightarrow f(x_1) < f(x_2)$

As we move from left to right

f is decreasing on I $\quad x_1 < x_2 \longrightarrow f(x_1) > f(x_2)$

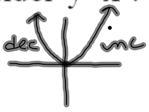
Functions with positive derivatives are increasing functions; functions with negative derivatives are decreasing functions.

Let f be continuous on [a,b] and differentiable on (a,b)

If $f' > 0$ at each point of (a,b), then f increases on [a,b]

If $f' < 0$ at each point of (a,b), then f decreases on [a,b]

Consider $y=x^2$. Where is it increasing and decreasing?



$y' = 2x \quad 0 = 2x \quad x = 0$

$(-\infty, 0) \;\; 0 \;\; (0, \infty)$

y': $-$ | $+$

y: dec | inc

Find local extrema, where increasing, where decreasing

#22 page 206 $\qquad y = x^4 - 10x^2 + 9 \qquad D: (-\infty, \infty)$

$y' = 4x^3 - 20x$

$y' = 4x(x^2 - 5)$

$0 = 4x$

$\boxed{0 = x}$

$(x - \sqrt{5})(x + \sqrt{5})$

$x - \sqrt{5} = 0 \qquad x + \sqrt{5} = 0$

$x = \sqrt{5} \qquad x = -\sqrt{5}$

min $\;\;$ max $\;\;$ min

$-\sqrt{5} \quad 0 \quad \sqrt{5}$

$(-\infty, -\sqrt{5}) \;\; (-\sqrt{5}, 0) \;\; (0, \sqrt{5}) \;\; (\sqrt{5}, \infty)$

y': $-$ | $+$ | $-$ | $+$

y: dec | inc | dec | inc

min $\underline{-16}$ at $x = -\sqrt{5}$ $\qquad \boxed{(-\sqrt{5})^4 - 10(-\sqrt{5})^2 + 9}$

min $\underline{-16}$ at $x = \sqrt{5}$ $\qquad \boxed{(\sqrt{5})^4 - 10(\sqrt{5})^2 + 9}$

max $\underline{9}$ at $x = 0$ $\qquad y(0) = 0^4 - 10(0)^2 + 9$

inc $(-\sqrt{5}, 0) \cup (\sqrt{5}, \infty)$

dec $(-\infty, -\sqrt{5}) \cup (0, \sqrt{5})$

To sum up so far if f′(c) =0 then (c, f(c)) is a possible max or min. One way to figure it out is to consider f′(x) on either side of c.
If f′(a) is positive, f(x) is increasing.
If f′(a) is negative, f(x) is decreasing.
What if f′(c) = 0 for all points in the domain?
Functions with f′ = 0 are constant functions.

function is y=c

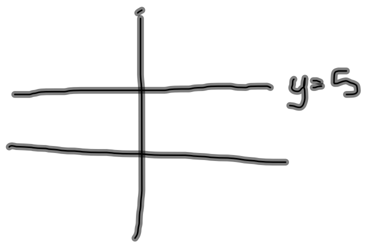
f′(c) = 0 for all c then

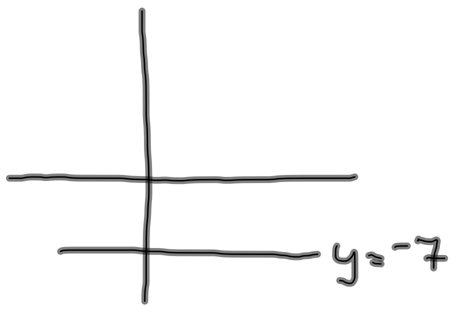
$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \qquad f(x_2) - f(x_1) = 0 \qquad so \qquad f(x_2) = f(x_1)$$

Example f(x) = 3 we know f′(x) = 0

f′(c) is always zero. $\dfrac{3-3}{10-3} = \dfrac{0}{7} = 0$

If f′(x) = 0 then $\boxed{f(x) = c}$





This next idea will prove to be very important over the next few months.
If functions have the same derivative, then the original functions differ by a constant.
If $f'(x) = g'(x)$ then $f(x) = g(x) + c$

Example: Find dy/dx
$y = 2x + 3$ $y' = 2$
$y = 2x - 4$ $y' = 2$
$y = 2x$ $y' = 2$
$y = 2x + 100$ $y' = 2$

Also d/dx
$y = \sin(x) + 2$ $y' = \cos(x)$
$y = \sin(x)$ $y' = \cos(x)$
$y = \sin(x) - 4$ $y' = \cos(x)$

Find all possible functions whose derivative is
Examples:

$f' = 3$ $f(x) = 3x + C$

$f' = x^1$ $f(x) = \frac{1}{2}x^2 + C$ $f'(x) = x^n$ $\frac{x^{n+1}}{n+1}$

$f' = 2x$ $f(x) = x^2 + C$

$f' = -\sin x$ $f(x) = \cos x + C$

$f' = \dfrac{1}{\sqrt{1-x^2}}$ $f(x) = \sin^{-1} x + C$

$f' = e^x + 2$ $f(x) = e^x + 2x + C$

$f' = 3x^2$ $f(x) = x^3 + C$

$f' = \sec^2 x$ $f(x) = \tan(x) + C$

$y' = 3x^2 - 2x + 4$ $y = x^3 - x^2 + 4x + C$

Look at #35 page 207

function has

$f'(x) = -\frac{1}{x^2}, \quad x > 0, \boxed{P(2,1)}$

$x_1 \; y_1$

$\frac{-1}{x^2} = -1x^{-2}$

$f'(x) = -x^{-2} \Rightarrow f(x) = x^{-2+1}$

$f(x) = \frac{-1x^{-2+1}}{(-2+1)}$

$\frac{-1x^{-1}}{-1} \Rightarrow x^{-1}$

guess x^{-1} $f(x) = x^{-1} + c$

when $x=1$ y or $f(1)=1$

$so\ 1 = (2)^{-1} + c \Rightarrow 1 = \frac{1}{2} + c \Rightarrow \frac{1}{2} = c$

$so\ \boxed{f(x) = \frac{1}{x} + \frac{1}{2}} \quad x > 0$

$y = \frac{1}{x} + C$

$1 = \frac{1}{2} + C$

$\frac{1}{2} = C$

Now we are finding the actual function instead of just the graph. We are applying the rules of differentiation in reverse. The name for this process is antidifferentiation.

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A function F(x) is an antiderivative of a function f(x) if f′(x) = f(x) for all x in the domain of f.

Given that acceleration due to gravity is 9.8 m/sec^2, find a function that represents the velocity and position.

$s = position$

$s' = v = velocity$

$s'' = a = acceleration$

$a = 9.8$

$v = 9.8t + c = 9.8t + v_0$

$s = 4.9t^2 + v_0 t + c = 4.9t^2 + v_0 t + s_0$

$a = 9.8$

$V = 9.8t + C$

initial velocity

$S = \frac{9.8t^2}{2} + V_0 t + C$

initial position

$\boxed{S = 4.9t^2 + V_0 t + S_0}$