4.4 notes calculus

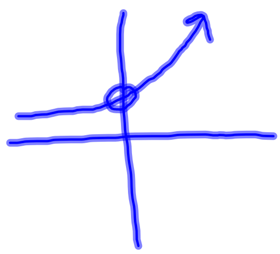
Derivatives of Exponential and Logarithmic Functions

When we reviewed exponentials and logarithms in chapter 1 we mentioned that logs and exponential with base *e* come up a great deal. *e* is extremely unique, not just because it shows up in nature; but for several other reasons. One we will examine now.

Recall $\lim_{x\to 0}\frac{\sin x}{x}=1$ we first showed this using a graph.

Consider $\lim_{h\to 0}\frac{e^h-1}{h}=1$ *Graph*

$y_1=\frac{(e^x-1)}{x}$



This means something remarkable when we find the derivative of e^x. $f(x)=e^x$ $f(x+h)=e^{x+h}$

$$\frac{d}{dx}(e^x)=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} \Rightarrow \frac{e^x\cdot e^h-e^x}{h} \Rightarrow \lim_{h\to 0}\frac{e^x(e^h-1)}{h}$$

$$=\lim_{h\to 0}e^x\frac{(e^h-1)}{h} \Rightarrow \boxed{\lim_{h\to 0}e^x}\lim_{h\to 0}\frac{(e^h-1)}{h}=e^x\cdot 1=\boxed{e^x}$$

$$\boxed{\frac{d}{dx}(e^x)=e^x}$$

The function is its own derivative!

In a general form we apply the chain rule and get

$$\boxed{\frac{d}{dx}(e^u)=e^u\cdot\frac{du}{dx}}$$

It would be nice to differentiate other exponential functions. Let's try a clever trick. Knowing

$$\frac{d}{dx}(e^x) = e^x \quad and \quad a^x = e^{x\ln a}$$

$e^{x\ln a} \Rightarrow e^{\ln a^x}$

$e^{\log_e \boxed{a^x}}$

$e^u \cdot \frac{du}{dx}$

$$\frac{d}{dx}(e^{x\ln a}) = e^{x\ln a}(\ln a) = e^{\ln a^x} \cdot \ln a = a^x \ln a$$

$u = \ln a \cdot x \qquad \frac{du}{dx} = \ln a$

$\ln a = \log_e a$

$\boxed{\text{constant}}$

$$\frac{d}{dx}a^x = a^x \cdot \ln a$$

$\frac{d}{dx}$ exponential function is exp function times ln(base)

$$In\ general: \frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$$

$$\frac{d}{dx}a^u = a^u \cdot \ln a \cdot \frac{du}{dx}$$

exp function ln base deriv of exponent

Examples:

$y = 2e^x \qquad \frac{dy}{dx} = 2e^x$

$y = e^{2x} \qquad \frac{d}{dx} = e^{2x} \cdot 2 \qquad \frac{d}{dx} = 2e^{2x}$

$e^u \cdot \frac{du}{dx}$

$u = 2x$

$\frac{du}{dx} = 2$

$y = 8^x \qquad y' = 8^x \ln 8$

$a^x \qquad a^x \cdot \ln(\text{base})$

$y = x^\pi$ power function $\qquad y' = \pi(x)^{\pi - 1} \qquad \frac{d}{dx} x^n = n \cdot x^{n-1}$

$y = 3^{\csc(x)} \qquad \boxed{y' = -3^{\csc(x)} \ln 3(\csc x \cot x)}$

exp function

$\frac{d}{dx}(3^u) = 3^u \cdot \ln 3 \cdot \frac{du}{dx} \qquad \frac{d}{dx}(\csc x)$

$3^{\csc x} \cdot \ln 3 \cdot (-\csc x \cdot \cot x)$

$2 \cdot 7^x = \cancel{\ln 49^x}$ $\cancel{14^x}$

$2 \cdot 7^2 \neq 14^2$

$2 \cdot 49 \neq 196$

$98 \neq 196$

7^x exponential function

x^7 power function

$\frac{d}{dx} x^7 = 7x^6$

$\frac{d}{dx}(7^x) = 7^x \cdot \ln 7$

$\ln 7 \cdot 7^x$

That takes care of exponentials, not that they are all that easy. What about logarithms. Let's do another clever move.

$y = \ln(x)$ $\quad e^{y} = e^{\ln(x)}$ $\quad e^{y} = x$

power base number

$\log_e x = y$ $\quad e^{y} = x$ $\quad e^{y} = e^{\ln(x)}$

$$\frac{d}{dx}e^{y} = \frac{d}{dx}x$$

$e^{y} = x$

find derivative implicit diff.

$$e^{y} \cdot \frac{dy}{dx} = 1$$

$$e^{y} \cdot \frac{dy}{dx} = 1$$

$$e^{y} \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^{y}} \Rightarrow \frac{1}{x}$$ $e^y = x$

$$\frac{dy}{dx} = \frac{1}{e^{y}}$$ but $e^{y} = x$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$so \quad \frac{dy}{dx}\ln(x) = \frac{1}{x}$$

* In general: $$\boxed{\frac{d}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx}}$$

$$\frac{d}{dx}\ln(x-3) \qquad \frac{1}{x-3} \cdot 1$$

$$\frac{d}{dx}\left[\ln(2x^3-3x+5)\right] = \frac{1}{2x^3-3x+5} \cdot 6x^2-3$$

$$\boxed{\frac{6x^2-3}{2x^3-3x+5}}$$

Derive a formula for $\frac{d}{dx}\log_a x$

How could this be rewritten with functions you already know?

Use Change of base formula: $\log_a x = \frac{\ln x}{\ln a}$

$$\frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{\ln a}\frac{d}{dx}\ln x = \frac{1}{\ln a}\cdot\frac{1}{x} = \frac{1}{x\ln a}$$

$$\boxed{\text{In general } \frac{du}{dx}\left(\log_a u\right) = \frac{1}{u\ln a}\frac{du}{dx}}$$

$$\frac{1}{\ln a}\cdot\frac{1}{u}\cdot\frac{du}{dx}$$

$\log_3(x^2+2x)$

rewrite $\frac{\ln(x^2+2x)}{\ln 3}$ find $\frac{d}{dx}\frac{\ln(x^2+2x)}{\ln 3}$

$$\frac{1}{\ln 3}\cdot\frac{1}{x^2+2x}\cdot 2x+2$$

$$\frac{2x+2}{\ln(3)\cdot(x^2+2x)}$$

Examples:

15. $y = \ln(x^2)$ $\quad y' = \frac{1}{x^2}2x = \frac{2}{x}$ $\quad \frac{1}{u}\cdot\frac{du}{dx}$

$\ln u$

16. $y = (\ln x)^2$ $\quad y' = 2\ln x \cdot \frac{1}{x}$ $\Rightarrow \frac{2\ln x}{x}$

$2(\ln x)^1 \cdot \frac{d}{dx}(\ln x)$

21. $y = \log_4 x^2$ $\quad y' = \frac{1}{x^2 \ln 4}2x = \frac{2}{x\ln 4}$

$\log_a u$ $\quad \frac{1}{u \ln a}\cdot\frac{du}{dx}$

24. $y = \frac{1}{\log_2 x}$

$$y' = (\log_2 x)^{-1} = -1(\log_2 x)^{-2}\frac{1}{x\ln 2} = \frac{-1}{(\log_2 x)^2 x\ln 2}$$

$$= \frac{-1}{x(\ln 2)(\log_2 x)^2} = \frac{-1\ln 2}{x(\ln x)^2}$$

24. $y = \frac{1}{\log_2 x}$ find y'

$(\log_2 x)^{-1}$

$-1(\log_2 x)^{-2} \cdot \frac{d}{dx}(\log_2 x)$

$\frac{-1}{(\log_2 x)^2}\cdot\frac{1}{x}\cdot\frac{1}{\ln 2}$

$\log_2 x = \frac{\ln x}{\ln 2}$

$$\frac{-1}{\left(x\,\cancel{\ln 2}\cdot\frac{\ln x}{\cancel{\ln 2}}\cdot\frac{\ln x}{\ln 2}\right)}$$

$\frac{-1}{1} \div \frac{x\cdot\ln x\cdot\ln x}{\ln 2}$

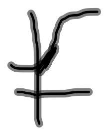
$\frac{-1}{1}\cdot\frac{\ln 2}{x(\ln x)^2}$

$\frac{-1\ln 2}{x(\ln x)^2}$

Exploration 1

p179

① $y(0)$ $t=0$ $y=72-30(.98)^0$ $72-30=\boxed{42^\circ F}$



② $\lim_{t\to\infty} \Rightarrow 72^\circ$

③ $y' = -30(.98)^t \ln .98$ $y'' = -30(\ln .98)^2(.98)^t$

$t=0$

④ $55 = 72-30(.98)^t$ $\frac{\ln\left(\frac{17}{30}\right)}{\ln .98} \approx 28.114\text{min}$

$-17 = -30(.98)^t$

$\frac{17}{30} = .98^t$ $y'(28.114) = -30(.98)^{28.114}(\ln .98)$

$\ln\frac{17}{30} = t\ln .98$

$.343^\circ F$ per min

Example 8

We need to revisit the power rule one more time

If u is a positive differentiable function and n is any real number then

$$\boxed{\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}}$$

Example 5

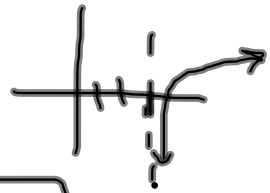
$y = x^{\sqrt{2}}$ $\quad \frac{dy}{dx} = \sqrt{2}\, x^{\sqrt{2}-1}$

$y = (2+\sin 3x)^{\pi}$ $\quad \pi(2+\sin 3x)^{\pi-1} \cdot \cos(3x) \cdot 3$

$3\pi \cos(3x) \cdot (2+\sin 3x)^{\pi-1}$

Example 6

$f(x) = \ln(x-3)$ Domain $x-3$ $x>3.$



$f'(x) = \frac{1}{x-3} \cdot 1$

$= \frac{1}{x-3}$ Domain $x \neq 3$ $\boxed{x>3}$

Example 7

$M = N$

$\ln M = \ln N$

$y = x^x$

$\ln y = \ln x^x$

implicit $\ln(y) = x \cdot \ln x$ product

$\frac{1}{y} \cdot \frac{dy}{dx} = \left(x \cdot \frac{1}{x}\right) + (\ln x)(1)$

$\frac{1}{y} \frac{dy}{dx} = 1 + \ln x$

MULT by y

$\frac{dy}{dx} = y(1+\ln x)$

$\frac{dy}{dx} = x^x(1+\ln x)$

Logarithmic Differentiation: do when both base and exponent are functions of x.

#47. $y = x^{\ln x}$

$\ln y = \ln x^{\ln x}$

$\ln(y) = \ln x \cdot \ln x$

$\frac{1}{y} \cdot \frac{dy}{dx} = (\ln x) \cdot \frac{1}{x} + (\ln x)\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = \frac{2\ln x}{x} \cdot y$

$\frac{dy}{dx} = \frac{x^{\ln x} \cdot 2\ln x}{x}$

OR

$\ln y = (\ln x)^2$

$\frac{1}{y}\frac{dy}{dx} = 2(\ln x)^1 \cdot \frac{1}{x}$

$\frac{dy}{dx} \cdot \frac{1}{y} = \frac{2\ln x}{x}$

$\frac{dy}{dx} = y \cdot \frac{2\ln x}{x}$

$\frac{dy}{dx} = x^{\ln x} \cdot \frac{2\ln x}{x}$