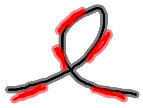
4.2 notes calculus

Implicit Differentiation

Look at figure 4.7 
What is the equation of this function? $x^3+y^3-9xy=0$
Does this graph have tangents?
How can we differentiate this function? NO CLUE YET $\frac{dy}{dx}$
On this particular example we would have to break it into three function pieces. These 3 pieces are not explicitly stated, they are defined implicitly. Therefore the process by which we find the derivative of such a function is called **Implicit differentiation**.
Implicit differentiation is used when y cannot be written explicitly as a function of x.
What does implicit and explicit mean? $y=$

SOLVE for y
don't know what y is in terms of x

Implicit differentiation – treat y as a differentiable function of *x* and apply the rules of differentiation.

Let's start with a surprisingly simple concept. What is the derivative of y with respect to x?
Answer: The derivative of y is $\boxed{dy/dx}$.

$y' = \frac{dy}{dx}$

Example:

$\boxed{Find\ \frac{dy}{dx}}$ $if\ y^2 = x$

$\frac{d}{dx}(y^2) = \frac{d}{dx}(x)$

$\frac{d}{dx}[(y)^2] = 1$

Use the chain rule:

$\frac{d}{dx}$ *outside* ~~*of inside*~~ leave inside alone *times deriv inside* $2y\frac{dy}{dx}=1$

blob²

Find $\frac{dy}{dx}$

$2(y)^1 \cdot \frac{dy}{dx} = 1$

Divide by 2y

$\frac{dy}{dx} = \frac{1}{2y}$

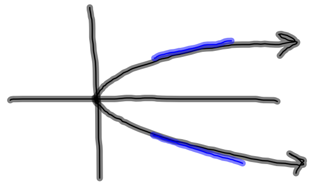
$\frac{dy}{dx} = \frac{1}{2y}$

dy/dx does not always have to be a function of x. Sometimes it is useful to have it be a function of y. Let's read the paragraph under Example 1.

$y^2 = x$

$y = \pm\sqrt{x}$

$\frac{dy}{dx} = \pm\frac{1}{2\sqrt{x}}$



$x^3 + y^3 - 9xy = 0$ Find $\frac{dy}{dx}$

$\frac{d}{dx}(x^3) + \frac{d}{dx}((y)^3) - 9\frac{d}{dx}(xy)$

blob³ product

$3x^2 + 3(y)^2 \cdot \frac{dy}{dx} - 9\left[x\frac{d}{dx}(y)^1 + y(1)\right] = 0$

$1(y)^0 \cdot \frac{dy}{dx}$

$3x^2 + 3y^2\frac{dy}{dx} - 9\left[x\frac{dy}{dx} + y\right] = 0$

$3x^2 + 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} - 9y = 0$

find solve for $\frac{dy}{dx}$

$3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = -3x^2 + 9y$

$\frac{dy}{dx}(3y^2 - 9x) = -3x^2 + 9y$

$$\boxed{\frac{dy}{dx} = \frac{-3x^2 + 9y}{3y^2 - 9x}}$$

Implicit Differentiation Process:

1. Differentiate both sides of the equation with respect to x.
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx.

Examples:

Find

$\frac{dy}{dx}$ *if* $2x^3 - 3(y)^2 = 8$

$$6x^2 - 6(y)\cdot\frac{dy}{dx} = 0$$

$$6x^2 = 6y\frac{dy}{dx} \qquad \frac{-6y\frac{dy}{dx}}{-6y} = \frac{-6x^2}{-6y}$$

$$\frac{6x^2}{6y} = \frac{dy}{dx}$$

$$\frac{x^2}{y} = \frac{dy}{dx} \qquad \boxed{\frac{dy}{dx} = \frac{x^2}{y}}$$

$\frac{d}{dx}$ *if* $x^2 + xy - y^2 = 1$

product

$$x^2 + x(y) - (y)^2 = 1$$

$$2x + \left[x\frac{dy}{dx} + y(1)\right] - 2(y)'\frac{dy}{dx} = 0$$

$$2x + x\frac{dy}{dx} + y - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x - 2y) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{x - 2y}$$

on multiple choice

$$\frac{-(2x+y)}{-(-x+2y)}$$

$$\frac{2x+y}{2y-x}$$

#18. Find $\frac{d}{dx}$ if $x^2 + y^2 = 25$

Find the lines that are (a) tangent and (b) normal to the curve at the given point. (3,-4)

$x_1 \quad y_1$

$$x^2 + (y)^2 = 25$$

$$2x + 2(y)^1 \frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} \qquad \frac{dy}{dx} = \frac{-x}{y}$$

EQ of tangent line

$$\left.\frac{dy}{dx}\right|_{(3,-4)} = \frac{-3}{-4}$$

$$y - y_1 = m(x - x_1)$$

$$\boxed{y + 4 = \frac{3}{4}(x - 3)}$$

EQ of Normal line

$$\boxed{y + 4 = -\frac{4}{3}(x - 3)}$$

We can also find higher order derivatives implicitly.

Example: $Find \dfrac{d^2y}{dx^2}$ of $2x^3 - 3y^2 = 8$

$\dfrac{dy}{dx}$

$$6x^2 - 3 \cdot 2y^1 \cdot \frac{dy}{dx} = 0$$

$$-6y\frac{dy}{dx} = -6x^2$$

$$\frac{dy}{dx} = \frac{-6x^2}{-6y} \qquad \frac{dy}{dx} = \frac{x^2}{y}$$

$\dfrac{d^2y}{dx^2}$

$$\frac{y \cdot 2x - x^2 \frac{dy}{dx}}{y^2}$$

$$\frac{2xy - x^2\left(\frac{x^2}{y}\right)}{y^2} \qquad \frac{\frac{y}{y}(2xy) - \frac{x^4}{y}}{y^2} = \frac{\frac{2xy^2 - x^4}{y}}{y^2}$$

$$= \boxed{\frac{2xy^2 - x^4}{y^3}}$$

Rational Powers of Differentiable Functions: We can apply the **power rule**,

As we have shown before: $\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{2\sqrt{x}}$

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{d}{dx}x^{\frac{1}{2}} \Rightarrow \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

Something more challenging?!

$Find \ \frac{dy}{dx} \quad if \quad y = (2x-3)^{\frac{2}{3}}$

$$\frac{dy}{dx} = \frac{2}{3}(2x-3)^{-\frac{1}{3}}$$

$$\frac{2}{3(2x-3)^{\frac{1}{3}}}$$

$$\frac{2}{3\sqrt[3]{2x-3}}$$

$Find \ \frac{dy}{dx} \quad if \quad y = \sqrt[5]{\sin(x)}$

$$y = (\sin x)^{\frac{1}{5}}$$

$$\frac{dy}{dx} = \frac{1}{5}(\sin x)^{-\frac{4}{5}}$$

$$\frac{1}{5(\sin x)^{\frac{4}{5}}}$$

$$\frac{1}{5\sqrt[5]{(\sin x)^4}} \text{ or } \frac{1}{5\sqrt[5]{\sin^4(x)}}$$

Exploration 1

$$x^2 + (-2x)(y) + (y)^2 = 4$$

product

① $2x + \left[-2x \cdot 1(\cancel{y})' \cdot \frac{dy}{dx} + y(-2)\right] + 2(y)^1 \cdot \frac{dy}{dx} = 0$

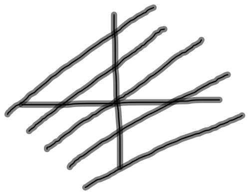
$$2x + -2x\frac{dy}{dx} + -2y + 2y\frac{dy}{dx} = 0$$

$$-2x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x + 2y$$

$$\frac{dy}{dx}(-2x + 2y) = -2x + 2y$$

$$\frac{dy}{dx} = \frac{-2x + 2y}{-2x + 2y} \qquad \frac{dy}{dx} = 1$$

② $\frac{dy}{dx} = 1 \qquad y = x + C$



③ Let $x = 0$

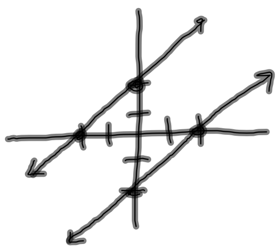
$x^2 - 2xy + y^2 = 4$

$\cancel{0^2} \cancel{-2(0)y} + y^2 = 4$

$y = \pm 2$

$y = x + c$
$2 = 0 + c$
$2 = c$
$\boxed{y = x + 2}$

$y = x + c$
$-2 = 0 + c$
$-2 = c$
$\boxed{y = x - 2}$



④ $x^2 - 2xy + y^2 = 4$

$(x - y)(x - y) = 4$

$(x - y)^2 = 4$

$x - y = \pm 2$

$x - y = 2$
$x - 2 = y$
$\boxed{y = x - 2}$

$x - y = -2$
$\boxed{y = x + 2}$