4.1 notes calc

Chain Rule

We know how to differentiate lots of functions, polynomials, trig functions, rational functions, etc., but we have not done derivatives of composites.

How do we differentiate $y=\sin(x^2+x)$?

We actually use a new rule for differentiation which is the ***most widely used rule in calculus***, the ***chain rule.***

Suppose we make an easy composite function. $y=3(x^2+4x)$
This could be made up of $y=3(u)$ and $u=x^2+4x$ so
$y=3x^2+12x$ $y(u)=3u$ $u(x)=x^2+4x$
$y'=6x+12$

$\frac{dy}{dx}=6x+12$ $\frac{dy}{du}=3$ $\frac{du}{dx}=2x+4$

$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ $\frac{dy}{du}=3$ $\frac{du}{dx}=2x+4$

$\frac{dy}{\cancel{du}}\cdot\frac{\cancel{du}}{dx}=\frac{dy}{dx}$

$3(2x+4)$
$\boxed{6x+12}$

Here is another example:
$y=9x^4+6x^2+1$ $(3x^2+1)^2$ $9x^4+6x^2+1$
$y=u^2$ and $u=3x^2+1$ u^2 $\sqrt{9x^4}$ $\sqrt{1}$
$dy/du=2u$ and $du/dx=6x$ $12ux$ $3x^2$ $\cdot$ 1
so $y'=2(3x^2+1)\cdot 6x=(6x^2+2)\cdot 6x=36x^3+12x$ Does this match with the derivative of $9x^4+6x^2+1$?
$36x^3+12x$

How can we write this as a rule that will be easier to work with? As it turns out, the previous rule is actually the notation that

$2u\cdot 6x$
$\frac{dy}{dx}=2u\cdot 6x$
$2(3x^2+1)\cdot 6x$
$(6x^2+2)6x$
$36x^3+12x$

Leibniz worked out, but we generally use a method that Newton developed that relates to typical composite notation.

The Chain Rule:
If f is differentiable at the point $u=g(x)$ and g is differentiable at x, then the composite function
$\boxed{(f\circ g)'(x)}=f'(g(x))\cdot g'(x)$
$\frac{d}{dx}\underbrace{f(g(x))}_{\text{composite}}=f'(g(x))\cdot g'(x)$

In Leibniz notation if $y=f(u)$ and $u=g(x)$, then $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$
where dy/du is evaluated at $u=g(x)$.

Remember this using the words "outside" and "inside".
$\boxed{\text{The derivative of the outside, leave the inside alone, times the derivative of the inside.}}$

CHAIN RULE

If $y = \sin(x^2+x)$
$f(x) = \sin(x)$ "outside" and $g(x) = x^2+x$ "inside"
Then y' = derivative of the outside function evaluated at the inside function left alone times the derivative of the inside

$y = \sin(x^2+x)$
BLOB
$y' = \cos(x^2+x)\cdot(2x+1)$ $(2x+1)\cos(x^2+x)$
$\frac{d}{dx}\sin(\text{BLOB})\cdot\frac{d}{dx}(\text{BLOB})$
inside inside

Another example:
$y = 2(3x - 5)$ outside: $y = 2u$ inside: $u = 3x - 5$
$6x-10$ $\frac{dy}{du} = 2$ $\frac{du}{dx} = 3$
$y' = 2$ $u' = 3$ y' of u times $u' = 2(3) = 6$.
Does this match with the derivative of $6x - 10$? 6
yes

After some practice, the rule becomes pretty easy to use.

Example Differentiate: $\sin(2x+1)$

Outside derivative $\cos(2x + 1)$
Inside derivative 2
Outside times inside: $2\cos(2x + 1)$

#3 $y = \cos(\sqrt{3}\cdot x)$
$\frac{d}{dx}$ outside $\to -\sin(\sqrt{3}\cdot x)$ $\frac{d}{dx}$ inside $\to \sqrt{3}$ $y' = -\sqrt{3}\sin(\sqrt{3}\cdot x)$
$-\sin(\sqrt{3}x)\cdot\sqrt{3}$
$-\sqrt{3}\sin(\sqrt{3}x)$

We can also apply the chain rule repeatedly
Consider:
$y = \sin^4(3x) = (\sin(3x))^4 \xrightarrow{dy/dx} 4(\sin(3x))^3\cdot\cos(3x)\cdot 3 = 12\sin^3(3x)\cos(3x)$

$y = (\sin(3x))^4$
$y' = 4(\sin 3x)^3\cdot\frac{d}{dx}(\sin 3x)$
$y' = 4(\sin 3x)^3\cdot\cos(3x)\cdot\frac{d}{dx}(3x)$
$y' = 4(\sin 3x)^3\cdot\cos(3x)\cdot 3$
$\boxed{12(\sin 3x)^3\cdot\cos(3x)}$

#22 $y = (1+\cos 2x)^2$
$2(1+\cos 2x)\,(-\sin 2x)\cdot 2 = -4(\sin 2x)(1+\cos 2x)$

$y = (1+\cos(2x))^2$
inside
$2(1+\cos(2x))^1\cdot\frac{d}{dx}(1+\cos 2x)$
$0+\frac{d}{dx}\cos(2x)$
$2(1+\cos(2x))^1\cdot -\sin(2x)\cdot\frac{d}{dx}(2x)$
$2(1+\cos(2x))^1\cdot -\sin(2x)\cdot 2$
$\boxed{-4(1+\cos(2x))^1\sin(2x)}$

Example 3

$y = \cos(t^2+1)$
inside
$\frac{dy}{dt} = -\sin(t^2+1)\cdot\frac{d}{dt}(t^2+1)$
$-\sin(t^2+1)\cdot 2t$
$\boxed{-2t\sin(t^2+1)}$

Slopes of Parametrized Curves

Parametric curves still have tangents so they should have derivatives. Using the chain rule we can find dy/dx parametrically.

If all three derivatives exist and $dx/dt \neq 0$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$x(t)$ $y(t)$ $a \le t \le b$

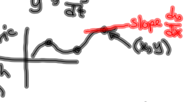
$x' = \frac{dx}{dt}$ $y' = \frac{dy}{dt}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$\frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dx}$

graph: parametric

just graph (x,y)



Example: say $x = \sin t$ $y = \cos t$ $0 \le t \le 2\pi$

$x(t) = \sin t$ $y(t) = \cos t$ $0 \le t \le 2\pi$

Find dy/dx.

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ $-\sin t \div \cos t$ $\frac{dy}{dt} \div \frac{dx}{dt}$

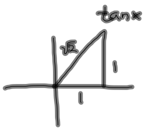
Find the slope of the tangent at $t = \pi/4$.

Find the equation of the normal line at $t = \pi/4$.

$\frac{dy}{dx} = \frac{-\sin t}{\cos t} \Rightarrow -\tan(t)$

$\tan x = \frac{y}{x}$ $\frac{opp}{adj}$

$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}}$ $-\tan\left(\frac{\pi}{4}\right)$



$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -1$

EQ of tangent line first at $t = \frac{\pi}{4}$

$y - y_1 = m(x - x_1)$ (x_1, y_1)

$\boxed{y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)}$

$x(t) = \sin t$

$x_1 = \sin\frac{\pi}{4} \Rightarrow \frac{\sqrt{2}}{2}$

$y = -x + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$ $y = -x + \sqrt{2}$ simplified

$y(t) = \cos t$

$y_1 = \cos\frac{\pi}{4} \Rightarrow \frac{\sqrt{2}}{2}$

EQUATION of Normal line

$$y - \frac{\sqrt{2}}{2} = 1\left(x - \frac{\sqrt{2}}{2}\right)$$

Simplified

$y = x - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$ $\boxed{y = x}$

Because powers are used so often and polynomials are so easy to differentiate, we have what's called "The Power Chain Rule". It is easy to see by examples.

Example

$y = \sin^2(3x)$ $(\sin(3x))^2$

$2(\sin(3x))' \cdot 3\cos(3x)$

$2u \cdot du/dx$

$y = (x^3 - 2x)^4$ u^4

$4(x^3 - 2x)^3 (3x^2 - 2)$

$4u^3 \cdot du/dx$

If we think of these as $u^n \rightarrow d/dx = nu^{n-1} \cdot du/dx$

$(\sin(3x))^2$

$2(\sin 3x)^1 \cdot \cos(3x) \cdot 3$

$6\sin(3x)\cdot\cos(3x)$

$4(x^3 - 2x)^3 \cdot (3x^2 - 2)$

#23

$y = (1 + \cos^2 7x)^3$

$3(1 + \cos^2 7x)^2 (2\cos 7x)(-\sin 7x)7$

$-42 (\sin 7x)(\cos 7x)(1 + \cos^2 7x)^2$

$y = (1 + \cos^2 7x)^3$ inside

$y' = 3(1+\cos^2 7x)^2 \cdot \frac{d}{dx}(1 + (\cos 7x)^2)$

$y' = 3(1+\cos^2 7x)^2 \cdot (0 + 2(\cos 7x)^1) \cdot \frac{d}{dx}(\cos 7x)$ outside

$y' = 3(1+\cos^2 7x)^2 (2\cos 7x)^1 (-\sin(7x)) \cdot \frac{d}{dx}(7x)$

$y' = 3(1+\cos^2 7x)^2 (2\cos(7x))(-\sin(7x)) \cdot 7$

$$\boxed{-42(1+\cos^2 7x)^2 (\cos(7x))\sin(7x)}$$

Reminder, we always use θ measured in radians. All the formulas only work for radians. Yes it can be done in degrees, but it is very complicated!!!!!!

Radians are linear (numbers)

degrees are circular (not numbers)

radians

$45° \cdot \frac{\pi}{180} = \frac{\pi}{4}$