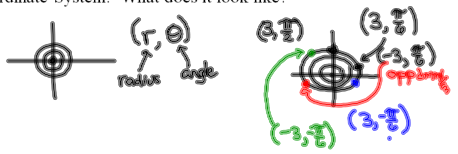
11.3 notes calculus

Polar coordinates and Polar Graphs

When we first learned about vectors we learned that vectors had magnitude and direction. We then had position vectors that told us how far the particle was from the origin and in which direction. As it turns out there is a coordinate system that is set up for this type of tracking system. It is called the Polar Coordinate System. What does it look like?



Polar Coordinates

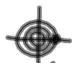
$P(r,\theta)$

r is the directed distance from O to P θ is the directed angle from initial ray to ray OP

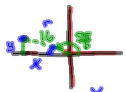
Example 1

① $(4, \frac{\pi}{2})$ ② $(-3, \pi)$ ③ $(16, \frac{5\pi}{6})$

(0,4)



(3,0)



$\cos\theta = \frac{x}{r}$ $\sin\theta = \frac{y}{r}$

$r\cos\theta = x$ $r\sin\theta = y$

$16\cos\frac{5\pi}{6}$ $16\sin\frac{5\pi}{6} = y$

$16(-\frac{\sqrt{3}}{2})$ $16(\frac{1}{2})$

$(-8\sqrt{3}, 8)$

④ $(-\sqrt{2}, -\frac{\pi}{4})$

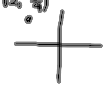
$x = r\cos\theta$ $y = r\sin\theta$

$x = -\sqrt{2}(\cos -\frac{\pi}{4})$ $y = -\sqrt{2}(\sin -\frac{\pi}{4})$

$x = -\sqrt{2} \cdot \frac{\sqrt{2}}{2}$ $y = -\sqrt{2}(-\frac{\sqrt{2}}{2})$

$x = -1$ $y = 1$

$(-\sqrt{2}, -\frac{\pi}{4})$ $(-1, 1)$



Polar Graphing

Equation Polar Graph

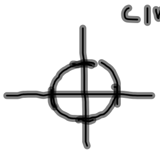
r = *a* Circle of radius |*a*| centered at 0.

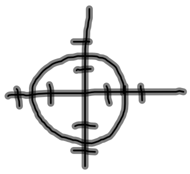
θ = *a* Line through O making an angle *a* with the initial ray.

Examples: r = 1 r =1.5

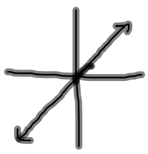
circle

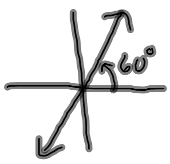
r = 1

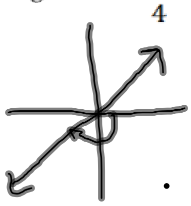




Examples: $\theta = \frac{\pi}{4}$ $\theta = \frac{\pi}{3}$ $\theta = -\frac{3\pi}{4}$

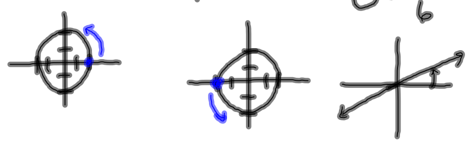






Example 2 Graphing with Polar Coordinates

$r=2$ $r=-2$ $\theta=\frac{\pi}{6}$



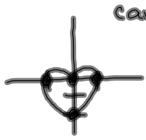
There are much more interesting polar functions

$r = a \pm b$ $a=b$

Example: r = 1 – sin θ (Graph by hand)

cardiod

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	1	0	1	2	1



rose with 12 petals

Example 3: r = sin 6θ r = 1 – 2cosθ r = 4sinθ

$r = a\sin(n\theta)$

$r = a\cos(n\theta)$

Rose

a determines the length of petal

n determines how many petals

if n is even 2·n petals

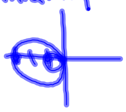
if n is odd n petals

$a \pm b\cos\theta$

$a \pm b\sin\theta$

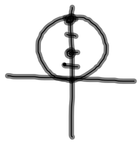
$a<b$

limacon with an inner loop

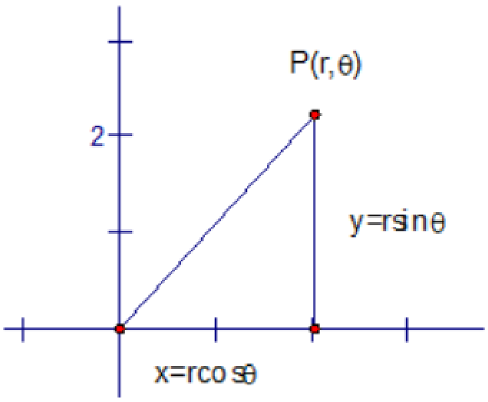


circle

sym with y



Without much difficulty we can show how polar graphs relate to Cartesian graphs. Polar graphs have a huge advantage in that they can graph things that don't appear to be functions in the Cartesian form. To make the conversion, we need only picture any point on a polar graph.



There are other relationships that naturally arise. Using the theorem of Pythagoras we get.

$\boxed{x = r\cos\theta}$ and $\boxed{y = r\sin\theta}$ then $\boxed{x^2 + y^2 = r^2}$

Also using trig we get $\tan\theta = \frac{y}{x}$ or $\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$

Example 4:

$r = 4\sin\theta$

$r^2 = 4r\sin\theta$ mult both sides by r

$x^2+y^2 = 4y$

$x^2+y^2-4y+\boxed{4} = 0+\boxed{4}$

$x^2+(y-2)^2 = 4$ $r=2$ $(0,2)$

$x^2+y^2=r^2$

$x = r\cos\theta$

$y = r\sin\theta$

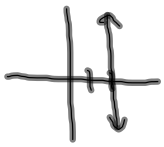
$r\cos\theta = 2$

$r^2\cos\theta\sin\theta = 4$

$r^2\cos^2\theta - r^2\sin^2\theta = 1$

$r\cos\theta = 2$

$x = 2$



$r^2\cos\theta\sin\theta = 4$

$r\cos\theta \cdot r\sin\theta = 4$

$xy = 4$

$y = \frac{4}{x}$

$r^2\cos^2\theta - r^2\sin^2\theta = 1$

$(r\cos\theta)^2 - (r\sin\theta)^2 = 1$

$x^2 - y^2 = 1$

The Parametric Equations of Polar Curves

The polar graph of r = f(θ) is the curve defined parametrically by:

x = rcos(θ) = f(θ) cos(θ) and y = rsin(θ) = f(θ) sin(θ)

Since polar curves are drawn in the xy-plane, the slope of a polar curve is still the slope of the tangent line, which is $\frac{dy}{dx}$. The polar-rectangular conversion formulas enable us to write x and y as functions of θ, so we can find $\frac{dy}{dx}$. $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$. If f is a differentiable function of θ, then so are x and y and, when dx/dθ ≠ 0, we can calculate dy/dx from the parametric formula.

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f(\theta)\cos\theta + \sin(\theta)\cdot f'(\theta)}{-f(\theta)\sin\theta + \cos\theta \cdot f'(\theta)}$$

Slope of the curve r = f(θ)

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

We should note that special circumstances **do** arise.

1. Horizontal tangent at a point where $dy/d\theta = 0$ and $dx/d\theta \neq 0$
2. Vertical tangent at a point where $dx/d\theta = 0$ and $dy/d\theta \neq 0$
3. Inconclusive when $dx/d\theta = dy/d\theta = 0$ $\Leftarrow$ L'Hopital to determine horizontal or vertical tangent

Example 5

$r = 2\sin 3\theta$ $\quad f(\theta) = 2\sin(3\theta)$

$f'(\theta) = 6\cos(3\theta)$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$\frac{dy}{dx}\Bigg| \frac{6\cos(3\theta)\sin\theta + 2\sin(3\theta)\cdot\cos\theta}{6\cos(3\theta)\cos\theta - 2\sin(3\theta)\cdot\sin\theta}$$

$$\frac{dy}{dx}\Bigg|_{\theta=\frac{\pi}{6}} \frac{\cancel{6\cos(3\cdot\frac{\pi}{6})\sin(\frac{\pi}{6})}+2\sin(3\cdot\frac{\pi}{6})\cos(\frac{\pi}{6})}{\cancel{6\cos(3\cdot\frac{\pi}{6})\cos(\frac{\pi}{6})}-2\sin(3\cdot\frac{\pi}{6})\sin(\frac{\pi}{6})}$$

$$\frac{\cancel{2}(1)\frac{\sqrt{3}}{\cancel{2}}}{-\cancel{2}(1)\frac{1}{\cancel{2}}} \quad \frac{\sqrt{3}}{-1}$$

$$\boxed{-\sqrt{3}}$$

39. Find the slope of the curve at each indicated point.
$r = -1 + \sin\theta, \theta = 0, \pi$

$f(\theta) = -1 + \sin\theta$

$f'(\theta) = \cos\theta$

$$\frac{dy}{dx}\Bigg|_{\theta=0} = \frac{\underset{1}{\cos(0)}\underset{0}{\cancel{\sin(0)}} + \underset{(-1+0)}{(-1+\cancel{\sin(0)})}\underset{(1)}{\cos(0)}}{\underset{1}{\cos(0)}\underset{1}{\cos(0)} - \cancel{(-1+\sin(0))\sin(0)}}$$

$(-1+0)\,(0)$

$$\frac{0 + -1(1)}{1 - 0} = \frac{-1}{1} = \boxed{-1}$$

$$\frac{dy}{dx}\Bigg|_{\theta=\pi} \frac{\cancel{\cos(\pi)\sin(\pi)} + (-1+\cancel{\sin\pi})\cos\pi}{\cos(\pi)\cos(\pi) - \cancel{(-1+\sin\pi)\sin\pi}}$$

$(-1)\,(0)$ $\quad$ 0 $\quad$ (-1)

$-1 \cdot -1$ $\quad$ $(-1+0)\,(0)$

$$\frac{(-1+0)(-1)}{1-0} \quad \frac{1}{1} = \boxed{1}$$

Area in Polar Coordinates: The area of the region between the origin and the **curve** $r = f(\theta)$ for $\alpha \le \theta \le \beta$ is $\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta} = \boxed{\int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 \, d\theta}$

Example 6

$$2\int_0^{\pi} \frac{1}{2}\left(2(1+\cos\theta)\right)^2 d\theta$$

$$\int_0^{\pi} (2+2\cos\theta)^2 \, d\theta$$

$$\int_0^{\pi} 4 + 8\cos\theta + 4\cos^2\theta \, d\theta$$

$$\int_0^{\pi} 4 + 8\cos\theta + 4\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta$$

$$\int_0^{\pi} 4 + 8\cos\theta + 2 + 2\cos 2\theta \, d\theta$$

$$\int_0^{\pi} 6 + 8\cos\theta + 2\cos 2\theta \, d\theta$$

$$6\theta + 8\sin\theta + \frac{2\sin(2\theta)}{2} \Big|_0^{\pi}$$

$$\left[6\pi + 8(0) + \sin(2\pi)\right] - \left[6(0) - 8(0) + \sin(0)\right]$$

$$\boxed{6\pi}$$

Example 7

inside smaller loop $\quad r = 2\cos\theta + 1$ $\quad$ $\frac{1}{2}$ is $120°$ to $180°$

$$2\int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2}(2\cos\theta+1)^2 \, d\theta$$

or

$$\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(2\cos\theta+1)^2 \, d\theta$$

$$\int_{\frac{2\pi}{3}}^{\pi} (2\cos\theta+1)^2 \, d\theta$$

$r_1 = 2\cos\theta + 1$

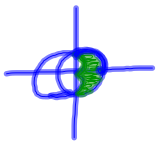
fnint$\left(r_1^2, \theta, \frac{2\pi}{3}, \pi\right) \approx \boxed{.544}$

Area Between Polar Curves: The area of the region between $r_1(\theta)$ and $r_2(\theta)$ for $\alpha \le \theta \le \beta$ is

$$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2} {r_2}^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} {r_1}^2 \, d\theta} = \boxed{\int_{\alpha}^{\beta} \frac{1}{2}({r_2}^2 - {r_1}^2) \, d\theta}$$

Example 8

$r = 1 \quad r = 1 - \cos\theta$



$$2\int_0^{\frac{\pi}{2}} \frac{1}{2} r_o^2 - \frac{1}{2} r_i^2 \, d\theta$$

$$2\int_0^{\frac{\pi}{2}} \frac{1}{2}(r_o^2 - r_i^2) \, d\theta$$

$$\int_0^{\frac{\pi}{2}} 1^2 - (1-\cos\theta)^2 \, d\theta$$

$$\int_0^{\frac{\pi}{2}} 1 - (1 - 2\cos\theta + \cos^2\theta) \, d\theta$$

$$\int_0^{\frac{\pi}{2}} \cancel{1} \cancel{-1} + 2\cos\theta - \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta$$

$$\int_0^{\frac{\pi}{2}} 2\cos\theta - \frac{1}{2} - \frac{\cos 2\theta}{2} \, d\theta$$

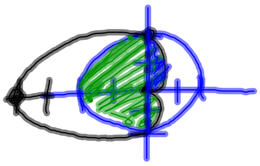
$$2\sin\theta - \frac{1}{2}\theta - \frac{\sin 2\theta}{4} \Big|_0^{\frac{\pi}{2}}$$

$$\left(2\sin\frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{\sin(\pi)}{4}\right) - (0)$$

$$\boxed{2 - \frac{\pi}{4}}$$

53. Find the area of the region shared by the circle r = 2 and the cardioid r = 2(1 – cosθ)

$r = 2 - 2\cos\theta$ $\int_a^b \frac{1}{2} r^2 \, d\theta$



part of cardiod

$\frac{1}{2}\text{circle} + 2\int_0^{\frac{\pi}{2}} \frac{1}{2}(2-2\cos\theta)^2 \, d\theta$

$\frac{1}{2}\pi r^2 + \int_0^{\frac{\pi}{2}} 4 - 8\cos\theta + 4\cos^2\theta \, d\theta$

$\frac{1}{2}\pi(2)^2$

$2\pi + \int_0^{\frac{\pi}{2}} 4 - 8\cos\theta + 4\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta$

$2\pi + \int_0^{\frac{\pi}{2}} 4 - 8\cos\theta + 2 + 2\cos 2\theta \, d\theta$

$2\pi + \left[6\theta - 8\sin\theta + \frac{\cancel{2}\sin 2\theta}{\cancel{2}}\Big|_0^{\frac{\pi}{2}}\right]$

$2\pi + \left[\left(6\left(\frac{\pi}{2}\right) - 8(1) \cancel{+0}\right) \cancel{-(0-0-0)}\right]$

$2\pi + 3\pi - 8$

$\boxed{5\pi - 8}$

function $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$

parametric $L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$

73. Length of a Polar Curve $\boxed{L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta}$

$x = r\cos\theta$ $\frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta$

$y = r\sin\theta$ $\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$

$\sqrt{\left(-r\sin\theta + \frac{dr}{d\theta}\cos\theta\right)^2 + \left(r\cos\theta + \frac{dr}{d\theta}\sin\theta\right)^2}$

$\sqrt{r^2\sin^2\theta \cancel{- 2r\frac{dr}{d\theta}\sin\theta\cos\theta} + \left(\frac{dr}{d\theta}\right)^2\cos^2\theta + r^2\cos^2\theta \cancel{+ 2r\cos\theta\frac{dr}{d\theta}\sin\theta} + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta}$

$\sqrt{r^2\sin^2\theta + r^2\cos^2\theta + \left(\frac{dr}{d\theta}\right)^2\cos^2\theta + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta}$

$\sqrt{r^2(\underbrace{\cancel{\sin^2\theta + \cos^2\theta}}_{1}) + \left(\frac{dr}{d\theta}\right)^2(\underbrace{\cancel{\cos^2\theta + \sin^2\theta}}_{1})}$

$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$