11.2 notes calculus

Vectors in the Plane

In some quantities we deal with, only the magnitude (value or number) is important. Your speedometer tells you how fast you are going. These are called scalars. If we are concerned about both the magnitude and direction, then we have a quantity called a vector. Scalar- only magnitude is important
Vector – magnitude and direction are important. There are many ways to represent a vector. We will learn several; the first is a geometric representation.

Definitions: Two-Dimensional Vector
A **two-dimensional vector v** is an ordered pair of real numbers, denoted in **component form** as $\langle a, b\rangle$. The numbers a and b are the **components** of the vector **v**. The **standard representation** of the vector
$\langle a, b\rangle$ is the arrow from the origin to the point (a,b). The **magnitude** (or **absolute value**) of **v**, denoted $|v|$, is **the length** of the arrow, and the **direction of v** is the direction in which the arrow is pointing. The vector **0** = $\langle 0, 0\rangle$, called the **zero vector**, has zero length and no direction.

Magnitude of a Vector:
The **magnitude** or **absolute value** of the vector $\langle a,b\rangle$ is the nonnegative real number $\boxed{|\langle a, b\rangle| = \sqrt{a^2 + b^2}}$.

Direction Angle of a Vector: The **direction angle** of a nonzero vector **v** is the smallest nonnegative angle **θ** formed with the **positive x-axis** as the initial ray and the standard representation of **v** as the terminal ray.

If an arrow has initial point (x_1,y_1) and terminal point (x_2,y_2), it represents the vector $\boxed{\langle x_2 - x_1, y_2 - y_1\rangle}$

Example 1

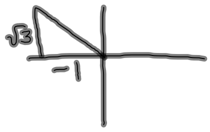
$V = \langle -1, \sqrt{3} \rangle$

magnitude

$|V| = \sqrt{(-1)^2 + (\sqrt{3})^2}$

$\sqrt{4}$

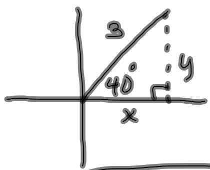
$\boxed{2}$



$\tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = -60°$ $\quad \tan^{-1}\left(\frac{y}{x}\right)$

$180 + -60 = \boxed{120°}$ $\quad \boxed{\frac{2\pi}{3}}$

Example 2



$\cos\theta = \frac{x}{r}$ $\quad \frac{adj}{hyp}$

$\boxed{r\cos\theta = x}$

$3\cos(40°)$

$\sin\theta = \frac{y}{r}$

$\sin\theta = \frac{opp}{hyp}$

$\boxed{r\sin\theta = y}$

$3\sin 40° = y$

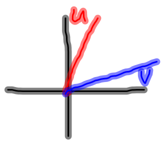
1.928

$\boxed{\langle 2.298,\ 1.928 \rangle}$

Definition: Vector Addition and Scalar Multiplication

Let $\boldsymbol{u} = \langle u_1, u_2 \rangle$ *and* $\boldsymbol{v} = \langle v_1, v_2 \rangle$ be vectors and let **k** be a real number (scalar).

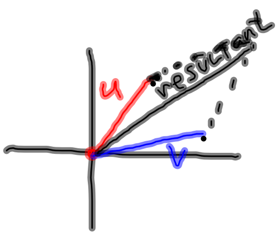
The **sum** or **resultant** of the vectors **u** and **v** is the vector $\boldsymbol{u} + \boldsymbol{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$

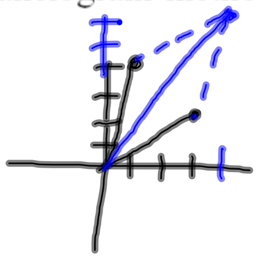


The **product of the scalar k** and the vector **u** is $k\boldsymbol{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle$

The **opposite of a vector v** is **–v = (-1)v**. We define vector subtraction by $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$

Whenever we add or subtract vectors the result is called the **resultant vector.** Do you remember the parallelogram method?





$\langle 1,4 \rangle + \langle 3,2 \rangle$

$\langle 4,6 \rangle$

Unit Vectors

u with length $|\mathbf{u}| = 1$ is a **unit vector.** If **v** is not the zero vector $\langle 0,0\rangle$, then the vector $u = \frac{v}{|v|}$ is a **unit vector in the direction of v.** Unit vectors provide a way to represent the direction of any nonzero vector. Any vector in the direction of **v**, or the opposite direction, is a scalar multiple of this unit vector **u**. The values correspond exactly to sin θ and cos θ.

In general $v = \langle a, b\rangle$; $a = |v|\cos\theta$ and $\mathbf{b} = |v|\sin\theta$

$x = |v|\cos\theta \qquad y = |v|\sin\theta$

$x = r\cos\theta \qquad y = r\sin\theta$

Example 3

(a) Let $u = \langle -1, 3\rangle \quad v = \langle 4, 7\rangle$

$2u + 3v$

$2\langle -1,3\rangle + 3\langle 4,7\rangle$

$\langle -2, 6\rangle + \langle 12, 21\rangle$

$\langle -2+12, 6+21\rangle$

$\boxed{\langle 10, 27\rangle}$

(b) $u - v$

$\langle -1,3\rangle - \langle 4,7\rangle$

$\langle -1-4, 3-7\rangle$

$\boxed{\langle -5, -4\rangle}$

(c) $\left|\frac{1}{2}u\right| = \frac{1}{2}\langle -1, 3\rangle$

$= \langle -\frac{1}{2}, \frac{3}{2}\rangle$

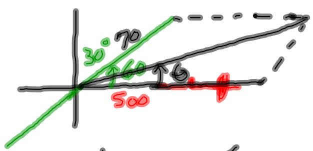
$\sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$

$\sqrt{\frac{1}{4} + \frac{9}{4}}$

$\sqrt{\frac{10}{4}} = \boxed{\frac{\sqrt{10}}{2}}$

Look at the Properties of Vector Operations on page 547.

Example 4 Finding Ground Speed and Direction



$x = r\cos\theta \qquad y = r\sin\theta.$

plane $\langle 500\cos 0°, 500\sin 0°\rangle$

wind $\langle 70\cos 60°, 70\sin 60°\rangle$

plane + wind $\langle 500\cos 0° + 70\cos 60°, 500\sin 0° + 70\sin 60°\rangle$

$\frac{1}{2} \qquad 500(0) \qquad 70\frac{\sqrt{3}}{2}$

$\langle 500 + 35, 0 + 35\sqrt{3}\rangle$

$p+w = \langle 535, 35\sqrt{3}\rangle$

ground speed $|p+w| = \sqrt{(535)^2 + (35\sqrt{3})^2}$

$\approx \boxed{538.424 \text{ mph}}$

$\theta = \tan^{-1}\left(\frac{35\sqrt{3}}{535}\right)$

$\approx 6.5°$ North of East

Example 5 Doing Calculus Componentwise

position $(\sin t, \frac{t^2}{2})$

a) position vector $\langle \sin t, \frac{t^2}{2} \rangle$

b) velocity $\langle \cos t, t \rangle$ take first derivative

c) $a(t)$ $\langle -\sin t, 1 \rangle$

d) Describe position and motion at $t=6$

Replace t with 6

position $\langle \sin 6, 18 \rangle$

velocity $\langle \cos 6, 6 \rangle$

acc $\langle -\sin 6, 1 \rangle$

moving right, away from origin

Velocity, Acceleration, and Speed

Suppose a particle moves along a smooth curve in the plane so that its position at any time t is (x(t), y(t)), where x and y are differentiable functions of t.

Particle's position vector is $r(t) = \langle x(t), y(t) \rangle$

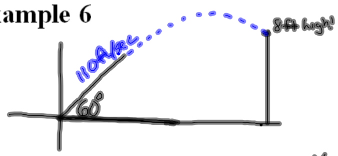
Particle's velocity vector is $v(t) = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle$

Particle's speed is the magnitude of **v**, denoted by $|v|$. Speed is scalar, not a vector. magnitude of velocity vector

Particle's acceleration vector is $a(t) = \langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \rangle$

Particle's direction of motion is the **direction vector** $\frac{v}{|v|}$

Example 6



a) $r(t) = \langle x(t), y(t) \rangle$ $x = (v_0 \cos\theta)t$ $y = -16t^2 + (v_0 \sin\theta)t$

a) $\langle (110\cos 60)t, -16t^2 + (110 \sin 60)t \rangle$

$\langle 55t, -16t^2 + 55\sqrt{3}t \rangle$

b) $v(t)$ $\langle 55, -32t + 55\sqrt{3} \rangle$

c) $8 = -16t^2 + 55\sqrt{3}t$
height $0 = -16t^2 + 55\sqrt{3}t - 8$

find zero $y = -16x^2 + 55\sqrt{3}x - 8$ zero at 5.869

$t \approx 5.869$ sec

horizontal distance

$x = 55t$ $55(5.869) \approx \boxed{322.780 \text{ ft}}$

d) speed velocity vector

$\sqrt{(55)^2 + (-32(5.869) + 55\sqrt{3})^2}$

107.655 ft/sec

Example 7

$r(t) = \langle \sin(3t), \cos(5t) \rangle$

$v(t) = \langle 3\cos(3t), -5\sin(5t) \rangle$

$a(t) = \langle -9\sin(3t), -25\cos(5t) \rangle$

Graph it ; Trace it

Example 8

position $(4\sin t, 2\cos t)$

a) $\vec{v}$ $\langle 4\cos t, -2\sin t \rangle$

$\vec{a}$ $\langle -4\sin t, -2\cos t \rangle$

b) $t = \frac{\pi}{4}$ $v(\frac{\pi}{4}) = \langle 4\cos\frac{\pi}{4}, -2\sin\frac{\pi}{4} \rangle$

$\langle 4 \cdot \frac{\sqrt{2}}{2}, -2\frac{\sqrt{2}}{2} \rangle$

$\langle 2\sqrt{2}, -\sqrt{2} \rangle$

$a(\frac{\pi}{4}) = \langle -4\sin\frac{\pi}{4}, -2\cos\frac{\pi}{4} \rangle$

$\langle -4(\frac{\sqrt{2}}{2}), -2(\frac{\sqrt{2}}{2}) \rangle$

$\langle -2\sqrt{2}, -\sqrt{2} \rangle$

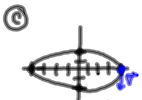
Speed $\sqrt{(2\sqrt{2})^2 + (-\sqrt{2})^2}$

$\sqrt{8+2}$ $\boxed{\sqrt{10}}$

direction of motion $\frac{v}{|v|}$ $\langle \frac{2\sqrt{2}}{\sqrt{10}}, \frac{-\sqrt{2}}{\sqrt{10}} \rangle$

or $\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle$

c)



$(\overset{x}{4\sin t}, \overset{y}{2\cos t})$

sketch path of particle

velocity vector at $(\overset{x}{4}, \overset{y}{0})$

$v(t) = \langle 4\cos t, -2\sin t \rangle$

position $\langle 4\sin t, 2\cos t \rangle$

$x = 4\sin t$ $y = 2\cos t$

$4 = 4\sin t$ $2\cos t = 0$

$\underline{1 = \sin t}$ $\cos t = 0$

clockwise

$v(t) = \langle 4\cos t, -2\sin t \rangle$

at point $(4,0)$

know $\cos t = 0$ and $\sin t = 1$

$\langle 4(0), -2(1) \rangle$

$\vec{v}$ $\langle 0, -2 \rangle$

Displacement and Distance Traveled

Suppose a particle moves along a path in the plane so that its velocity at any time t is

$\boldsymbol{v}(t) = (v_1(t), v_2(t))$, v_1 and v_2 are integrable functions of t.

The **displacement** from t = a to t = b is given by the vector

$\langle \int_a^b v_1(t)dt, \int_a^b v_2(t)dt \rangle$

The preceding vector is added to the position at time t = a to get the position at time t = b.

The **distance traveled** from t= a to t = b is

$\int_a^b |v(t)| dt = \int_a^b \sqrt{(v_1(t))^2 + (v_2(t))^2}\, dt$

Example 9

$v(t) = \langle t - 3\pi\cos\pi t, 2t - \pi\sin\pi t\rangle$

at $t=0$ particle is at point $(1,5)$

ⓐ $1 + \int_0^4 t - 3\pi\cos\pi t\, dt$ $\quad 5 + \int_0^4 2t - \pi\sin\pi t\, dt$

final position $\langle 9, 21\rangle$

b) $\int_0^4 \sqrt{(t - 3\pi\cos\pi t)^2 + (2t - \pi\sin\pi t)^2}\, dt$

≈ 33.533

$y_1 = t - 3\pi\cos(\pi t)$

$y_2 = 2t - \pi\sin(\pi t)$

$y_3 = y_1^2 + y_2^2$

fnint $(\sqrt{y_3}, x, 0, 4)$

Example 10

path from $(1,5)$ to $(9,21)$

$\frac{dx}{dt} = t - 3\pi\cos\pi t$ $\qquad \frac{dy}{dt} = 2t - \pi\sin\pi t$

$x = \int t - 3\pi\cos\pi t\, dt$ $\qquad y = \int 2t - \pi\sin\pi t\, dt$

$x = \frac{t^2}{2} - \cancel{\frac{3\pi\sin(\pi t)}{\pi}} + C$ $\qquad y = t^2 + \cos\pi t + C$

when $t=0$ $(1,5)$ $\quad x = \frac{t^2}{2} - 3\sin\pi t + C$ $\qquad 5 = 0^2 + \cos(\pi(0)) + C$

$1 = \frac{0^2}{2} - 3\sin(\pi \cdot 0) + C$ $\qquad 5 = 1 + C$

$1 = C$ $\qquad 4 = C$

$\boxed{x = \frac{t^2}{2} - 3\sin\pi t + 1}$ $\qquad \boxed{y = t^2 + \cos\pi t + 4}$