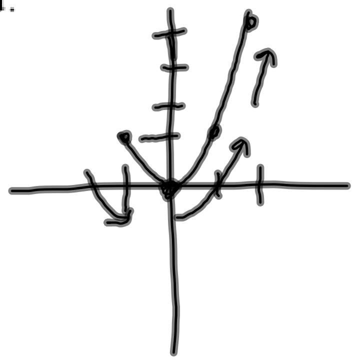
11.1 notes calculus

## Parametric Functions

This chapter will be about different ways to model motion and different coordinate systems that can be used to model many different situations. Some of this will seem like review and some will seem like pre-calculus rather than calculus.

Recall that a parameter is something that helps determine the setting of a function. In chapter 1 we used $x(t) = t$ and $y(t) = t^2$ to model a parabola parametrically. This basically means that both the x- and y- coordinates are determined by an outside parameter that isn't even graphed.

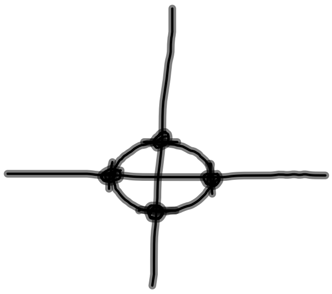
| $t$ | $x$ ($t$) | $y$ ($t^2$) |
|---|---|---|
| -1 | ( -1 | 1 ) |
| 0 | ( 0 | 0 ) |
| 1 | ( 1 | 1 ) |
| 2 | ( 2 | 4 ) |



# Example 1 Reviewing Some Parametric Curves

a) $x = \cos t \quad y = \sin t \quad [0, 2\pi)$

| $t$ | $x$ | $y$ |
|---|---|---|
| 0 | 1 | 0 |
| $\frac{\pi}{2}$ | 0 | 1 |
| $\pi$ | -1 | 0 |
| $\frac{3\pi}{2}$ | 0 | -1 |
| $2\pi$ | 1 | 0 |



$x^2 = \cos^2 t \quad y^2 = \sin^2 t$

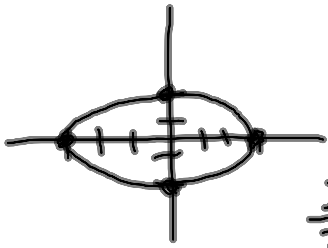
$x^2 + y^2 = \cos^2 t + \sin^2 t$

$x^2 + y^2 = 1$

circle

b) $x = 3\cos t \quad y = 2\sin t \quad [0, 4\pi]$ go around twice

| $t$ | $x$ | $y$ |
|---|---|---|
| 0 | 3 | 0 |
| $\frac{\pi}{2}$ | 0 | 2 |
| $\pi$ | -3 | 0 |
| $\frac{3\pi}{2}$ | 0 | -2 |
| $2\pi$ | -3 | 0 |



Ellipse

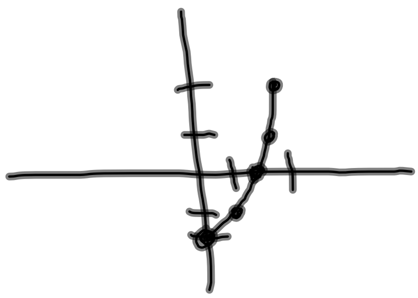
$\frac{x}{3} = \cos t \quad \frac{y}{2} = \sin t$

$\frac{x^2}{9} + \frac{y^2}{4} = \cos^2 t + \sin^2 t$

$$\boxed{\frac{x^2}{9} + \frac{y^2}{4} = 1}$$

c) $x = \sqrt{t} \quad y = t - 2 \quad [0, 4]$

| $t$ | $x$ | $y$ |
|---|---|---|
| 0 | 0 | -2 |
| 1 | 1 | -1 |
| 2 | $\sqrt{2}$ | 0 |
| 3 | $\sqrt{3}$ | 1 |
| 4 | 2 | 2 |



$(x)^2 = (\sqrt{t})^2 \quad y = x^2 - 2$ parabola

$x^2 = t \quad$ Domain $[0, 2]$

**Slope and Concavity**

Parametric Differentiation Formulas

If x and y are both differentiable functions of t and if $\frac{dx}{dt} \neq 0$,

then $\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$ If $y' = \frac{dy}{dx}$ is also a differentiable function of t,

then $\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}y' = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}}$

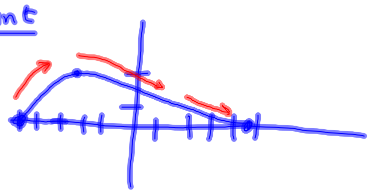
$$\frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}}$$

**Example 2**

$x = t^2 - 5 \quad y = 2\sin t \quad 0 \le t \le \pi$

a)

| t | x | 2sin t |
|---|---|---|
| 0 | -5 | 0 |
| $\frac{\pi}{2}$ | $\frac{\pi^2}{4} - 5$ | 2 |
| $\pi$ | $\pi^2 - 5$ | 0 |



$t = \pi/2$

$(-5, 0)$ $(-2.532599, 2)$

$(4.8696044, 0)$

b) highest point max $\frac{dy}{dx} = 0$

$\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\frac{2\cos t}{2t}$ set = 0 $\frac{2\cos t}{2t} = 0$

highest point occurs at $t = \frac{\pi}{2}$

$\boxed{(-2.532599, 2)}$

$2\cos t = 0$ $[0, \pi]$

$t = \frac{\pi}{2}$

@0 max $\pi/2$ @$\pi$

$\frac{2\cos t}{2t}$: $(0, \frac{\pi}{2})$ + | $(\frac{\pi}{2}, \pi)$ −

③ points of inflection

set $\frac{d^2y}{dx^2} = 0$ $\frac{d}{dt}\left(\frac{dy}{dx}\right)$ $\frac{d}{dt}\left(\frac{\cancel{2}\cos t}{\cancel{2}t}\right)$

$$\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{t(-\sin t) - \cos t(1)}{t^2}}{2t}$$

stay change flip

$$\frac{-t\sin t - \cos t}{t^2} \cdot \frac{1}{2t}$$

$$\frac{-t\sin t - \cos t}{2t^3} = 0$$ pt of inflection

$y_1 = -x\sin x - \cos x$ Find zero

pt of inflection occurs when $t = 2.798386$

Go back to parametric mode value for t

$\boxed{(2.831, 0.673)}$

10. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x = \frac{1}{t}$ y = -2 + ln(t)

$\frac{dx}{dt} = \frac{-1}{t^2}$ $\frac{dy}{dt} = \frac{1}{t}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{t}}{\frac{-1}{t^2}}$$

simplify

$$\frac{1}{t} \cdot \frac{-t^2}{1} = \boxed{-t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(-t)}{\frac{dx}{dt}} = \frac{-1}{\frac{-1}{t^2}}$$

$$\frac{-1}{1} \cdot \frac{-t^2}{1} = \boxed{t^2}$$

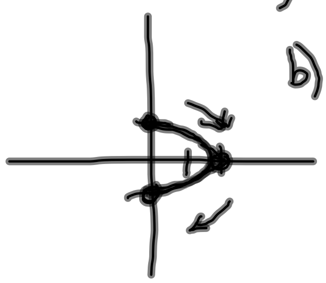
19. $x = 2\sin(t)\,; y = \cos(t) \quad for\ 0 \le t \le \pi$ Find rightmost point.

$\frac{dx}{dt} = 2\cos t \qquad \frac{dy}{dt} = -\sin t$

| $t$ | $x$ | $y$ |
|---|---|---|
| $0$ | $0$ | $1$ |
| $\frac{\pi}{2}$ | $2$ | $0$ |
| $\pi$ | $0$ | $-1$ |

a) graph it

b) $(2,0)$



c) max pt for $x$

set $\frac{dx}{dt} = 0$

$2\cos t = 0$

$\cos t = 0$

$t = \frac{\pi}{2}$

max $\frac{\pi}{2}$

| | $(0, \pi/2)$ | $(\frac{\pi}{2}, \pi)$ |
|---|---|---|
| $2\cos t$ | $+$ | $-$ |

Remember Length of a function $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

**Arc Length of a Smooth Parametrized Curve**

Let L be the length of a parametric curve that is traversed exactly once as t increases from $t_1$ to $t_2$. If $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous functions, then $\boxed{L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$

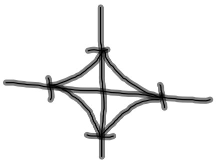
Remember that some parametric curves can act in strange ways and back track on itself.

## Example 3

Find the length of the ~~asteroid~~ astroid.

$x = \cos^3(t) \quad y = \sin^3(t) \quad 0 \le t \le 2\pi$

$x = (\cos t)^3 \quad y = (\sin t)^3$



$\frac{dx}{dt} = 3(\cos t)^2(-\sin t)$

$\left(\frac{dx}{dt}\right)^2 = 9(\cos t)^4 \sin^2 t$

$\frac{dy}{dt} = 3(\sin t)^2(\cos t)$

$\left(\frac{dy}{dt}\right)^2 = 9(\sin t)^4 \cos^2 t$

$4\int_0^{\frac{\pi}{2}} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\;dt$

$4\int_0^{\frac{\pi}{2}} \sqrt{9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)}\;dt$

$4\int_0^{\frac{\pi}{2}} \sqrt{9\cos^2 t \sin^2 t}\;dt$

$4\int_0^{\frac{\pi}{2}} 3\cos t \sin t\;dt$

$u = \sin t$

$du = \cos t\,dt$

$4\int_{u(0)}^{u(\frac{\pi}{2})} 3u\;du$

$4\left(\frac{3u^2}{2}\Big|_0^1\right)$

$4\left(\frac{3}{2} - 0\right)$

$\boxed{6}$

Find the length of the curve.

29. $x = 8\cos(t) + \underbrace{8t\sin(t)}$, $y = 8\sin(t) - 8t\cos(t)$, $0 \le t \le \pi/2$

$\frac{dx}{dt} = -8\sin t + 8t\cos t + 8\sin t$

$\frac{dy}{dt} = 8\cos t - [8t(-\sin t) + 8\cos t]$

$\frac{dx}{dt} = 8t\cos t$

$\frac{dy}{dt} = \cancel{8\cos t} + 8t\sin t - \cancel{8\cos t}$

$\left(\frac{dx}{dt}\right)^2 = 64t^2\cos^2 t$

$\left(\frac{dy}{dt}\right)^2 = 64t^2\sin^2 t$

$$\int_0^{\frac{\pi}{2}} \sqrt{64t^2\cos^2 t + 64t^2\sin^2 t}\, dt$$

$$\int_0^{\frac{\pi}{2}} \sqrt{64t^2(\cos^2 t + \sin^2 t)}\, dt$$

$$\int_0^{\frac{\pi}{2}} \sqrt{64t^2}\, dt$$

$$\int_0^{\frac{\pi}{2}} 8t\, dt$$

$$\frac{8t^2}{2}$$

$$4t^2 \Big|_0^{\frac{\pi}{2}}$$

$$4\left(\frac{\pi}{2}\right)^2 - 4(0)^2$$

$$4\left(\frac{\pi^2}{4}\right)$$

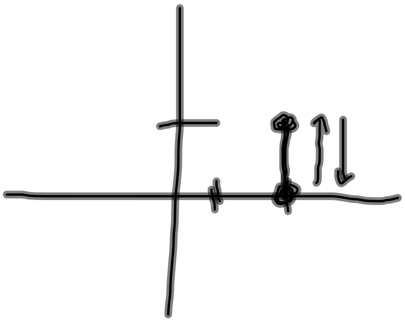
$$\boxed{\pi^2}$$

Many of these work out nicely, but occasionally the result may be non-integrable. In such a case, use fnint. To show why the graph can't retrace itself, do the following:

**Example:** $x = 2$ $y = \sin(t)$ $0 < t < \pi$

$x = 2$ $y = \sin(t)$

| $t$ | $x$ | $y$ |
|---|---|---|
| $0$ | $2$ | $0$ |
| $\frac{\pi}{2}$ | $2$ | $1$ |
| $\pi$ | $2$ | $0$ |



$$\int_0^{\pi} \sqrt{0^2 + \cos^2 t}\, dt$$

$$\int_0^{\pi} \sqrt{\cos^2 t}\, dt$$

$$\int_0^{\pi} \cos t\, dt$$

$$\sin t \Big|_0^{\pi}$$

$$\sin \pi - \sin 0 = 0!$$

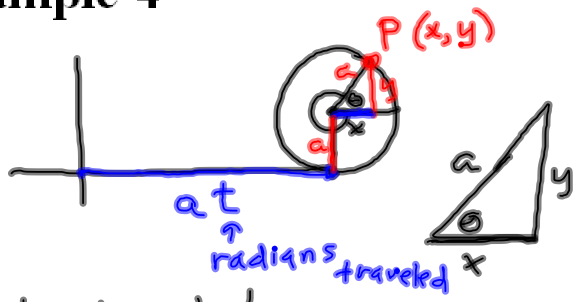
Length is Not even 0!

An interesting application for length of a curve is finding the length of a cycloid. A cycloid is a circular (oval) shaped graph that is periodic. Two examples are given in the book: A pendulum clock and a point on a wheel. The short of it is the graph is determined by parametric equations, so they are a good candidate for finding the length of a curve. You can look at the example on page 539 if you have a problem dealing with cycloids.

**Example 4**

$C = 2\pi r$

↑ radius $= a$




$\cos\theta = \frac{x}{a}$ $\quad \sin\theta = \frac{y}{a}$

$a\cos\theta = x$ $\quad a\sin\theta = y$

horizontal

$x = at + a\cos\theta$

height vertical

$y = a + a\sin\theta$

$t + \theta = \frac{3\pi}{2} + 2\pi k$

$\theta = \frac{3\pi}{2} - t + 2\pi k$

period $\cos\theta = 2\pi$

$x = at + a\cos\left(\frac{3\pi}{2} - t \cancel{+ 2\pi k}\right)$ coterminal

$at + a\cos\left(\frac{3\pi}{2} - t\right)$

$at + a\left[\cos\frac{3\pi}{2}\cos t + \sin\frac{3\pi}{2}\sin t\right]$

$a\left[\cancel{0\cos t} + {}^{-}1\sin t\right]$

$\boxed{x = at - a\sin t}$

$y = a + a\sin\left[\left(\frac{3\pi}{2} - t\right) \cancel{+2\pi k}\right]$

$a + a\left[\sin\frac{3\pi}{2}\cos t - \cos\frac{3\pi}{2}\sin t\right)$

$-1\cos t \qquad - \cancel{0\sin t}$

$\boxed{y = a - a\cos t}$

# Example 5

Find length of arch of cycloid

$x = at - a\sin t \quad y = a - a\cos t$

$\frac{dx}{dt} = a - a\cos t$

$\frac{dy}{dt} = a\sin t$

$$\int_0^{2\pi} \sqrt{(a - a\cos t)^2 + (a\sin t)^2}\, dt$$

$$\int_0^{2\pi} \sqrt{a^2 - 2a^2\cos t + a^2\cos^2 t + a^2\sin^2 t}\, dt$$

$$\sqrt{a^2(1 - 2\cos t + \underbrace{\cancel{\cos^2 t + \sin^2 t}}_{1})}\, dt$$

$$a\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

fnint $(\sqrt{2-2\cos x}, x, 0, 2\pi)$

$\boxed{8a}$