

10.5 notes calculus

### Testing Convergence at End Points

Without even being aware of it we already learned part of this section. In Exploration 1 of the last section, we used integrals to

show that  $\frac{1}{n}$  diverges and  $\frac{1}{n^2}$  converges. We showed this by using an area approximation. Look at page 512 figure 10.14 and page 517 figure 10.15

$$\int_1^{\infty} \frac{1}{x^p} \quad \begin{array}{l} \text{if } p \leq 1 \text{ diverge} \\ p > 1 \text{ converges} \end{array}$$

The idea is that if the integral converges or diverges, that will tell us something about the series. We swing the rectangles forward so the integral represents as upper bound. We swing the rectangles back so the function represents a lower bound.

#### Theorem 10: The Integral Test:

Let  $\{a_n\}$  be a sequence of positive terms. Suppose that  $a_n = f(n)$ , where  $f$  is a continuous, positive, decreasing function of  $x$  for all

$x \geq N$  ( $N$  a positive integer). Then the series  $\sum_{n=N}^{\infty} a_n$  and the

integral  $\int_N^{\infty} f(x) dx$  either both converge or both diverge.

$$\text{Ex } \sum_{n=1}^{\infty} \frac{1}{n} \quad \int_1^{\infty} \frac{1}{n} \quad \lim_{b \rightarrow \infty} \ln n \Big|_1^b$$

$\ln b - \ln 1$   
diverges

It really comes down to using a convergent or divergent integral for comparison. It is also important to note that the value of the integral may or may not be the same as the sum of the series.

Example 1:

Does  $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$  converge?

$$\int_1^{\infty} \frac{1}{n^{\frac{3}{2}}} dn$$

$$\lim_{b \rightarrow \infty} \int_1^b n^{-\frac{3}{2}} dn$$

$$\lim_{b \rightarrow \infty} -2n^{\frac{1}{2}} \Big|_1^b$$

$$\lim_{b \rightarrow \infty} \frac{-2}{\sqrt{b}} - \frac{-2}{\sqrt{1}} \rightarrow \text{Converges}$$

by the integral test  $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$  converges.

This leads directly into another category of series. the p-series.

Exploration 1

$\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if  $p > 1$

$$\int_1^{\infty} \frac{1}{n^p} \rightarrow \lim_{b \rightarrow \infty} \int_1^b n^{-p} \rightarrow \lim_{b \rightarrow \infty} \frac{n^{-p+1} b}{-p+1} \Big|_1^b$$

$$\lim_{b \rightarrow \infty} \frac{b^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}$$

$p > 1$  neg exp

$\sum_{n=1}^{\infty} \frac{1}{n^p}$  diverges if  $p \leq 1$

$$\int_1^{\infty} \frac{1}{n} dn \rightarrow \ln n$$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$  diverges if  $p < 1$

$$\frac{-1}{-p+1} = \frac{1}{p-1}$$

$$\frac{-1}{-(p-1)} = \frac{1}{p-1}$$

if  $p < 1$   
positive exponent  
 $b \rightarrow \infty$   
 $b^{-p+1} \rightarrow \infty$

Any series of the form  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is called a p-series. There are 3 cases:

1.  $p > 1$ , the series converges
2.  $p = 1$ , the series diverges
3.  $p < 1$ , the series diverges

We've just done an example where  $p > 1$ . We used  $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ .

If we tried  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  using the integral test, we see it diverges.

Finally, if  $p = 1$  we have  $\sum_{n=1}^{\infty} \frac{1}{n}$  which we know diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \rightarrow \int_1^{\infty} n^{-\frac{1}{2}} dn$$

$$\lim_{b \rightarrow \infty} \int_1^b n^{-\frac{1}{2}} dn$$

$$\lim_{b \rightarrow \infty} 2n^{\frac{1}{2}} \Big|_1^b$$

$$\lim_{b \rightarrow \infty} 2\sqrt{b} - 2\sqrt{1}$$

diverges

Because this is the border between the  $p$ -series that converge and diverge, it is of particular interest. It is also of interest because this series of numbers comes up occasionally in nature and in music.

The series  $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$  is called the **harmonic series**.

A few fascinating things about the harmonic series is

1) Because it diverges the sum must be infinite, but it grows

incredibly slow.  $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \ln n$

2) If  $p$  increased just slightly, it would converge. Even though series seldom come exactly in this form  $p$ -series are extremely useful.

We can use the simplicity of the  $p$ -series for comparison and we open up a whole new tool of functions/series for comparison.

### Theorem 11: The Limit Comparison Test (LCT)

Suppose that  $a_n > 0$  and  $b_n > 0$  for all  $n \geq N$  ( $N$  a positive integer)

1.  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$ ,  $0 < c < \infty$ , then  $\sum a_n$  and  $\sum b_n$  both converge or both diverge. This means that they both grow at the same rate, so they either must converge or diverge.

2.  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ , and  $\sum b_n$  converges, then  $\sum a_n$  converges.

This means that  $a_n$  grows slower than  $b_n$ , so if  $b_n$  converges so must  $a_n$ .

3.  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ , and  $\sum b_n$  diverges, then  $\sum a_n$  diverges. This means that  $a_n$  grows faster than  $b_n$ , so if  $b_n$  diverges so must  $a_n$ .

**Example 3 LCT**

a)  $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$  looks like  $\frac{2n}{n^2} = \frac{2}{n}$   
 $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges

$\lim_{n \rightarrow \infty} \frac{2n+1}{(n+1)^2} = \frac{2}{1} = 2$

b)  $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$  Looks like  $\frac{1}{2^n} = \left(\frac{1}{2}\right)^n$  geometric  
 $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$  converges

$\lim_{n \rightarrow \infty} \frac{1}{2^n - 1} \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$


c)  $\sum_{n=2}^{\infty} \frac{3n+2}{n^3 - 2n}$  looks like  $\frac{1}{n^2}$  integral test or p-series  
 Converges

$\lim_{n \rightarrow \infty} \frac{3n+2}{n^3 - 2n} = 0$

d)  $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$   $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges

$\lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$

by LCT  $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$  diverges



16.  $\sum_{n=1}^{\infty} \frac{5n^3 - 3n}{n^2(n+2)(n^2+5)}$  looks like  $\frac{5n^3}{n^5}$

$\sum_{n=1}^{\infty} \frac{5}{n^2}$  converges

$\lim_{n \rightarrow \infty} \frac{5n^3 - 3n}{n^2(n+2)(n^2+5)} = \frac{1}{n^2}$

$\lim_{n \rightarrow \infty} \frac{(5n^3 - 3n)n^2}{n^2(n+2)(n^2+5)} = \frac{5n^5 - 3n^3}{n^2(n+2)(n^2+5)} = 5$

by LCT and by p-series test  $\sum_{n=1}^{\infty} \frac{5n^3 - 3n}{n^2(n+2)(n^2+5)}$  converges

Recall that we have dealt with series of the form  $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ .

In this case the signs of the consecutive terms alternated between positive and negative. It is therefore called an alternating series. Alternating series have some interesting properties.

### Example:

The alternating harmonic series  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots + \frac{(-1)^{n+1}}{n} + \dots$

This converges.

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots + \frac{(-1)^n 4}{2^n} + \dots \quad r = -\frac{1}{2} \quad |r| < 1$$

This is a geometric series and converges to  $-4/3$ .

$$1 - 2 + 3 - 4 + 5 - 6 + \dots + (-1)^{n+1} n + \dots$$

This series diverges by the nth-Term Test.

### Theorem 12: The Alternating Series Test (Leibniz's Theorem)

The series  $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$  converges if all three of the following conditions are satisfied.

1. Each  $u_n$  is positive.

②  $u_n \geq u_{n+1}$  for all  $n \geq N$  for some integer  $N$ ;  $n \rightarrow \infty$

③  $\lim_{n \rightarrow \infty} u_n \rightarrow 0$

Look at figure 10.18.

We can use this picture to get an idea of how and why the series converges. We can also get another very important fact: what the error is.

**Theorem 13: The Alternating Series Estimation Theorem**

If the alternating series  $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$  satisfies the conditions of Theorem 12, then the truncation error for the  $n$ th partial sum is less than  $u_{n+1}$  and has the same sign as the unused term

Look at **Example 4**

**Read:** A note on the error bound

A few other interesting facts about absolutely convergent series:

- 1) Rearrangements of absolutely convergent series are also absolutely convergent
- 2) Rearrangements of conditionally convergent series can be made to diverge or converge to a predetermined sum.

Read the paragraph under the box “Rearrangements of conditionally Convergent Series.

**Example 5**

*Read it*

To put this section in a nutshell, we want to determine the radius of convergence and see if the series converges at the endpoints. There are three possibilities at each endpoint. 1) diverge, 2) absolutely converges, 3) conditionally converges.

You might remember that this is similar to how we started radius of convergence. 1) converges only at the center(  $x = a$ ); 2) converges everywhere ( $x = a$ ); 3) converges only on interval about the center.

The tough part of all of this will be keeping the tests straight and knowing what to do first. We will need to be organized and we will need lots of practice!

### How to Test a Power Series $\sum_{n=0}^{\infty} c_n(x-a)^n$ for convergence

1. Use the Ratio Test to find the values of  $x$  for which the series converges absolutely.

Ordinarily, this is an open interval.  $a - R < x < a + R$

In some instances, the series converges for all values of  $x$ . In rare cases, the series converges only at  $x = a$ .

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. The Ratio Test fails at these points. Use a comparison test, the Integral Test, or the Alternating Series Test.

3. If the interval of absolute convergence is  $a - R < x < a + R$ , conclude that the series diverges (it does not even converge conditionally) for  $|x - a| > R$ , because for those values of  $x$  the  $n$ th term does not approach zero.

36.  $\sum_{n=0}^{\infty} (x+5)^n$  *geometric*

$|x+5| < 1$   
 $-1 < x+5 < 1$   
 $-6 < x < -4$

$\sum_{n=0}^{\infty} (6+5)^n$   $\sum_{n=0}^{\infty} (-1)^n$  *diverges*  
 $\sum_{n=0}^{\infty} (-4+5)^n$   $\sum_{n=0}^{\infty} \frac{1+(-1)^n}{1+1}$  *diverges*

a)  $(-6, -4)$  *after test endpoints*  
 b)  $(-6, -4)$   
 c) none

38.  $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

$\lim_{n \rightarrow \infty} \frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n} \Rightarrow 3x-2$

$|3x-2| < 1$   
 $-1 < 3x-2 < 1$   
 $1 < 3x < 3$   
 $\frac{1}{3} < x < 1$

*Test endpoints*  
 $\sum_{n=1}^{\infty} \frac{(3(\frac{1}{3})-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  *ALT Harm. Conv*  
 $\sum_{n=1}^{\infty} \frac{(3 \cdot 1 - 2)^n}{n} = \sum_{n=1}^{\infty} \frac{1^n}{n}$  *Harm diverges*

a)  $[\frac{1}{3}, 1)$   
 b)  $(\frac{1}{3}, 1)$   
 c)  $x = \frac{1}{3}$

Example 6:

a)  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{2n}$

$\lim_{n \rightarrow \infty} \left| \frac{x^{2(n+1)}}{2(n+1)} \cdot \frac{2n}{x^{2n}} \right|$   
 $\lim_{n \rightarrow \infty} \frac{x^{2n+2}}{2n+2} \cdot \frac{2n}{x^{2n}} \Rightarrow x^2$   
 Converges if  $|x^2| < 1$

a)  $R=1$  *Test endpoints*  
 b)  $[-1, 1]$   
 c)  $(-1, 1)$  *converge*  
 d)  $x = \pm 1$  *Must test harm*

b)  $\sum_{n=0}^{\infty} \frac{(0x)^n}{n!}$

$\lim_{n \rightarrow \infty} \left| \frac{(0x)^{n+1}}{(n+1)!} \cdot \frac{n!}{(0x)^n} \right|$   
 $\lim_{n \rightarrow \infty} \frac{(0x)^{n+1}}{n+1} = 0 \quad n \rightarrow \infty$

a)  $R = \infty$  b)  $(-\infty, \infty)$  c)  $(-\infty, \infty)$  d) none

c)  $\sum_{n=0}^{\infty} n!(x+1)^n$

$\lim_{n \rightarrow \infty} \frac{(n+1)!(x+1)^{n+1}}{n!(x+1)^n} = (n+1)(x+1)$   
 $\lim_{n \rightarrow \infty} (n+1)(x+1)$  *diverge*

a)  $R=0$  b)  $[-1]$  c)  $[-1, 1]$  d) none

d)  $\sum_{n=1}^{\infty} \frac{(x-3)^n}{2n}$

$\lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{2(n+1)} \cdot \frac{2n}{(x-3)^n} \right| \Rightarrow x-3$

$|x-3| < 1$  *converge*  
 $-1 < x-3 < 1$   
 $2 < x < 4$

a)  $R=1$  b)  $(2, 4)$  c)  $(2, 4)$   
 d)  $x=2$  *Test endpoints*  
 $\sum_{n=1}^{\infty} \frac{(4-3)^n}{2n}$  *harmonic diverges*  
 $\sum_{n=1}^{\infty} \frac{(2-3)^n}{2n}$  *converge alt harm*



Look at page 526 Procedure for Determining Convergence

35.  $\sum_{n=0}^{\infty} x^n$

Geometric  
 $|x| < 1$

Test endpoints

$\sum_{n=0}^{\infty} 1^n$  diverge

$\sum_{n=0}^{\infty} (-1)^n$  diverge  
 $1 - 1 + 1$

- (b)  $(-1, 1)$  abs conv
- (a)  $(-1, 1)$  test end pts
- c) none conditional converge

7.  $\sum_{n=1}^{\infty} \frac{5}{n+1}$

Ratio test  $\left| \frac{5}{(n+1)+1} \cdot \frac{n+1}{5} \right| = 1$   
 inconclusive!

looks like  $\frac{1}{n}$

LCT  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges

$\lim_{n \rightarrow \infty} \frac{\frac{5}{n+1}}{\frac{1}{n}} = \frac{5}{n+1} \cdot n$

$\lim_{n \rightarrow \infty} \frac{5n}{n+1} \Rightarrow 5$

by LCT  $\sum_{n=1}^{\infty} \frac{5}{n+1}$  diverges