10.4 notes calculus

Radius of Convergence

Let's take a closer look at what actually makes a series work. The real question is: "what are the rules for working with a series and when we can use them." If a series converges then it represents a number, so we can treat it like a number. All the familiar operations and rules apply. If it doesn't converge then we have very few tools to work with.

$$\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$

Remember that

$\dfrac{\sin x}{\cos x} = \tan x$, *that* $x^2 - 1 = (x-1)(x+1)$, *and that* $\dfrac{x^2-1}{x-1} = x+1$.

These are all identities. One side can be substituted for the other. However, in the last example there is one time when they are not equivalent. This is also true with the series we use to represent functions.

Example 1 Consider the sentence

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots + \ldots + (-1)^n x^{2n} + \ldots$$

$\frac{a}{1-r}$ $a=1$ $r=-x^2$

For what values of x is this an identity?

$|r|<1$ $\quad |-x^2|<1$ $\quad -1<x^2<1$

$|x^2|<1$ $\quad -1<x^2$ always

$x^2<1$

$x^2-1<0$

$(x-1)(x+1)<0$

$\boxed{(-1,1)}$

This holds only for the interval of convergence.
We can see this graphically, we can apply reason to it (r > -1, r < 1) and because it's geometric, we have an analytic proof of convergence.

What if it weren't geometric?

We need some strategies for dealing with series that are not familiar and we still need to determine when they converge and can be treated as functions or numbers. So far, we have developed a few series and their intervals of convergence.

$\mathbb{R}$ ⇐ $\sin x$; $\cos x$; e^x ; $\ln(1+x)$ $(-1,1]$ $\tan^{-1}(x)$ $[-1,1]$

We can put these power series in three categories.

$\frac{1}{1-x}$ $\frac{1}{1+x}$ geometric $|r|<1$ $(-1,1)$

Theorem 5: The Convergence Theorem for Power Series.

1. There are three possibilities for $\sum_{n=0}^{\infty} c_n(x-a)^n$ with respect to convergence:

There is a positive number R such that the series diverges for $|x-a|>R$ but converges for $|x-a|<R$.

The series may or may not converge at either of the endpoints **x = *a* -R** and **x = *a* +R**

(Converges on a finite interval about the center)

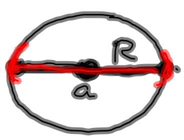
2. The series converges for every x (R = ∞)

(Converges on an infinite interval about the center)

3. The series converges at x = a and diverges elsewhere (R =0)

(Converges only at center)

The capital letter R is used for a reason. Because our series are developed around a center (Taylor x = a) we can figure out how far on either side they converge…like a radius. Therefore, this distance is called the **Radius of convergence**.



(So the number R is the radius of convergence **and** the set of all values of x for which the series converges is the interval of convergence.)

Example: $\sum_{n=0}^{\infty} 2^n x^n$ $\quad -\frac{1}{2} < x < \frac{1}{2} \to |x| < \frac{1}{2} \quad R = \frac{1}{2}$

$\sum_{n=0}^{\infty} (2x)^n$ $\quad |r|<1 \quad |2x|<1 \quad |x|<\frac{1}{2}$

$|2x|<1$

n=0 $\quad 1$
n=1 $\quad 2x \quad$ geometric
n=2 $\quad (2x)^2$

$-\frac{1}{2} < \frac{2x}{2} < \frac{1}{2}$

$-\frac{1}{2} < x < \frac{1}{2} \quad (-\frac{1}{2}, \frac{1}{2})$

$-\frac{1}{2}$ R $\frac{1}{2}$

Example: $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ $\quad |x|<1 \quad -1 < x < 1 \quad \boxed{R=1}$

$\sum_{n=0}^{\infty} (-1)^n (x^2)^n$ $\quad (-1x^2)^n$

n=0 $\quad 1$
n=1 $\quad -1x^2$
n=2 $\quad 1x^4$

$r = -x^2$

$|-x^2|<1$

$|x^2|<1$

$-1 < \boxed{x^2} < 1$

always true

-1 1

$x^2 < 1$

$(-1, 1)$
did it earlier

We'll figure out the endpoints in section 10.5.

Right now we learn how to determine the radius. The easiest thing to figure out is if the series diverges because the terms never get small.

The easiest thing to figure out is if the series diverges because the terms never get small. We saw the series 1+2+3+4.... at the beginning of the chapter.

Theorem 6: The nth Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ *diverges if* $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Only use this test to prove divergence!
Unless the n^{th} term of the series approaches "0" a series cannot converge.

29. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$\lim_{n\to\infty} \frac{n}{n+1} \quad \frac{\infty}{\infty} \quad$ L'Hopital

$\lim_{n\to\infty} \frac{1}{1} \Rightarrow 1$

$1 \neq 0$

diverge by n^{th} term test!

Recall the sandwich theorem for functions used in chapter 2. We used two well known functions with limits to sandwich another function thereby determine its limit. This is the first tool we will use to determine convergence or divergence of a new series.

Theorem 7: The Direct comparison Test DCT

Let $\sum a_n$ be a series with no negative terms.

$\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \le c_n$ for all n > N, for some integer N.

$\sum a_n$ diverges if there is a divergent series $\sum d_n$ of nonnegative terms with $a_n \ge d_n$ for all n > N, for some integer N.

We just need to show that beyond a certain point the new series is always above or below a divergent or convergent series.

To use this effectively, we need to have a good handle on common series that converge and diverge. 1/n ($\frac{1}{n}$) diverges, $\frac{1}{2^n}$ converges and they need to be non-negative.

Example 2: $\sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2}$ $\quad \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2$

$n=0 \quad 1$

$n=1 \quad x^2$

$n=2 \quad \frac{x^4}{(2!)^2}$

$$\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right) = e^x$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{x^{2n}}{(n!)^2} \le \frac{x^{2n}}{n!} \quad \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$$

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!}$$

e^x converges and e^{x^2} converges

By DCT $\sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2}$ converges.

33. $\sum_{n=1}^{\infty} \frac{2^n}{3^n+1}$

Looks like $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

Converges because geometric $\left|\frac{2}{3}\right| < 1$

$$\frac{2^n}{3^n+1} < \left(\frac{2}{3}\right)^n$$

by DCT $\sum_{n=1}^{\infty} \frac{2^n}{3^n+1}$ converges

In general, it can be difficult to use the comparison test for convergence because you have to know a series that converges and it must be easily compared. However, comparison is useful especially if it jumps out at you.

Sometimes we might know of a good comparison, but the series is alternating or negative.

Example:

$\sum_{n=1}^{\infty} \frac{-1}{2^n}$ Always negative

$$\sum_{n=1}^{\infty} (-1) \cdot \frac{1}{2^n}$$

$(-1) \sum_{n=1}^{\infty} \frac{1}{2^n}$ $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ geometric converges

Converges to $\frac{-\frac{1}{2}}{1-\frac{1}{2}} \Rightarrow \frac{-\frac{1}{2}}{\frac{1}{2}} = -1$

This leads to the idea of absolute convergence.

Definition: Absolute convergence

If the series $\sum |a_n|$ of absolute values converges, then $\sum a_n$ converges absolutely.

Theorem 8: Absolute Convergence Implies Convergence

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

The worst thing that could happen is that all of the terms could be negative. If this happened, we could just factor a negative out. If only some of them were negative, we take our answer and "subtract" those terms. It would still converge.

Show that this series converges for all x

Example 3: $\displaystyle\sum_{n=0}^{\infty} \frac{(\sin x)^n}{n!}$

$$\sum_{n=0}^{\infty} \frac{|\sin x|^n}{n!}$$

$$\frac{|\sin x|^n}{n!} \leq \frac{1}{n!}$$

know that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e^1$

since $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ replace x with 1

by DCT $\displaystyle\sum_{n=0}^{\infty} \frac{\sin x}{n!}$ converges for all reals.

The next thing we will discuss is a powerful test to determine convergence and also figure out the radius of convergence for an arbitrary power series. $\boxed{\text{It is called the ratio test.}}$

Theorem 9: The Ratio Test

Let $\sum a_n$ be a series with positive terms, and with $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

then

a. The series converges if $L < 1$

b. The series diverges if $L > 1$

c. The test is inconclusive if $L = 1$

hink about what this is saying: If this limit is bigger than 1, hen the "next" term is some fraction bigger than the previous erm. There is no way the series can converge because the terms re getting larger.

If the limit is less than one the "next" term is some fraction smaller than the previous term. This means that it would behave similar to a geometric series with r<1 (r = ratio between the terms)

Oddly enough, if r = 1, we get no information.

You might want to be aware that L'Hopital's rule may be useful in evaluating this limit if it's in indeterminate form.

Exploration 1

$\sum_{n=1}^{\infty} \frac{1}{n}$ $\boxed{\text{diverges}}$

$\lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right|$

$\lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1}$

$\lim_{n\to\infty} \frac{n}{n+1}$

$\lim_{n\to\infty} \frac{1}{1} = 1$

$L = 1$

$\int_1^{\infty} \frac{1}{n}$

$\ln n \Big|_1^{\infty}$

$\ln \infty - \ln 1$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}$

$\lim_{n\to\infty} \frac{1}{(n+1)^2} \cdot \frac{n^2}{1}$

$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} \Rightarrow 1$

$L = 1$

$\int_1^{\infty} \frac{1}{n^2}$ $\boxed{\text{Converge}}$

$\int_1^{\infty} n^{-2}$

$\frac{n^{-1}}{-1} \Big|_1^{\infty}$

$\frac{-1}{n} \Big|_1^{\infty}$

$\cancel{\frac{-1}{\infty}}^{0} - \frac{-1}{1}$

$\boxed{1}$

35. Determine if $\sum_{n=1}^{\infty} n^2 e^{-n}$ converges.

Ratio test

$\lim_{n\to\infty} \frac{(n+1)^2 e^{-(n+1)}}{n^2 e^{-n}}$

$\lim_{n\to\infty} \frac{(n+1)^2 e^{-n-1}}{n^2 e^{-n}}$

$\lim_{n\to\infty} \frac{(n+1)^2 \cancel{e^{-n}} \cdot e^{-1}}{n^2 \cancel{e^{-n}}}$

LHopital

$\lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \lim_{n\to\infty} e^{-1}$

$L = 1 \; e^{-1}$

Converges to e^{-1}

36. $\sum_{n=0}^{\infty} \frac{n^{10}}{10^n}$

Ratio test $\lim_{n\to\infty} \left| \frac{\frac{(n+1)^{10}}{10^{n+1}}}{\frac{n^{10}}{10^n}} \right|$

$\lim_{n\to\infty} \left| \frac{(n+1)^{10}}{\cancel{10^n}\cdot 10^1} \cdot \frac{\cancel{10^n}}{n^{10}} \right|$

$\lim_{n\to\infty} \frac{(n+1)^{10}}{n^{10}} \cdot \lim_{n\to\infty} \frac{1}{10^1}$

$\lim_{n\to\infty} 1 \cdot \frac{1}{10^1}$ $\boxed{\text{Converges } \frac{1}{10}}$

With problem 36 out of the way, let's add a slight twist.

Example 4 $\sum_{n=0}^{\infty} \frac{nx^n}{10^n}$

Ratio test $\lim_{n\to\infty} \left| \frac{\frac{(n+1)x^{n+1}}{10^{n+1}}}{\frac{nx^n}{10^n}} \right|$

$\lim_{n\to\infty} \frac{(n+1)\cancel{x^n}\cdot x^1}{\cancel{10^n}\cdot 10^1} \cdot \frac{\cancel{10^n}}{n\cdot \cancel{x^n}}$

$\lim_{n\to\infty} \frac{(n+1)}{n} \cdot \lim_{n\to\infty} \frac{x}{10}$

$\lim_{n\to\infty} 1 \cdot \frac{x}{10}$

Diverge if $\left|\frac{x}{10}\right| \geq 1$ converge if $\left|\frac{x}{10}\right| < 1$

Converge $-1 < \frac{x}{10} < 1$

Interval of convergence $-10 < x < 10$ $(-10, 10)$

$R = 10$

10. $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

Ratio test

$\lim_{n\to\infty} \left| \frac{\cancel{(3x-2)^{n+1}}\,(3x-2)^1}{(n+1)} \cdot \frac{n}{\cancel{(3x-2)^n}} \right|$

$\lim_{n\to\infty} \frac{n}{n+1} \cdot (3x-2)$

$3x-2$

Converge if $|3x-2| < 1$

$-1 < 3x-2 < 1$

$+2 \qquad +2$

$\frac{1}{3} < \frac{3x}{3} < \frac{3}{3}$

$\frac{1}{3} < x < 1$

$\boxed{\text{Interval of Convergence } \left(\frac{1}{3}, 1\right) \quad R = \frac{1}{3}}$

Occasionally we will run across a unique type of series that is not geometric and yet we can determine the exact value of the infinite sum.

Example 7

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$1 = A(n+1) + Bn$

$n=-1 \quad 1 = A(-1+1) + B(-1)$

$-1 = B$

$n=0 \quad 1 = A(1)$

$A = 1$

$\sum_{n=1}^{\infty} \frac{1}{n} + \frac{-1}{n+1}$

$n=1 \quad S_1 = 1 + \frac{-1}{2} = \frac{1}{2}$

$n=2 \quad S_2 = (1 + \cancel{\frac{-1}{2}}) + (\cancel{\frac{1}{2}} + \frac{-1}{3})$

$n=3 \quad S_3 = (1 + \cancel{\frac{-1}{2}}) + (\cancel{\frac{1}{2}} + \cancel{\frac{-1}{3}}) + (\cancel{\frac{1}{3}} + \frac{-1}{4})$

$\lim_{n\to\infty} S_n = 1 + \frac{-1}{n+1}$

$\boxed{\lim_{n\to\infty} 1}$

This is called a **collapsing** or **telescoping** series.

49. $$\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$$

$\frac{6}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$

$6 = A(2n+1) + B(2n-1)$

Let $n = \frac{-1}{2} \quad 6 = B(-2)$

$-3 = B$

Let $n = \frac{1}{2} \quad 6 = A(2) \quad A = 3$

$\sum_{n=1}^{\infty} \frac{3}{2n-1} + \frac{-3}{2n+1}$

$S_1 = \frac{3}{1} + \frac{-3}{3}$

$S_2 = \left(\frac{3}{1} + \frac{-3}{3}\right) + \left(\frac{3}{3} + \frac{-3}{5}\right)$

$S_3 = \left(\frac{3}{1} + \frac{-3}{3}\right) + \left(\frac{3}{3} + \frac{-3}{5}\right) + \left(\frac{3}{5} + \frac{-3}{7}\right)$

$S_n = \lim_{n\to\infty} 3 + \cancel{\frac{-3}{2n+1}} \to 0$

$\boxed{3}$

Part of the assignment is to complete Exploration 2 on page 513 in preparation for 10.5