10.3 notes calculus

Taylor's Theorem

With all the work that we have done with series and as interesting as some of the properties are, we still might ask," why would we want to use a series when we can use the actual function?"

The answer is that sometimes finding the exact answer can be impossible and it's more efficient to use an approximation. Besides, if you need $e^{4.2}$ (66.68633104) you use your calculator. It's just that you see an approximation that is correct to 8 or 10 decimal places. You don't see, or care about the error that has been $\boxed{\text{truncated}}$. This is really what this section is about:

1) Using the nth order Taylor polynomials to get an approximation.
2) Knowing what the associated error is.

cut off

Example of truncated

2.3876592

2.388 (rounded)

2.387 (truncated)

Example 1: Find a Taylor Polynomial that will serve as an adequate substitute for sin x on [-π, π]

centered at 0 $\sin\pi = 0 \quad \sin(-\pi) = 0$

There are a couple of issues here: We don't want to punch in an infinite number of terms, so it will be an approximation. We want it to be close, so we need to know what close enough (the error) is. Clearly, we should center the series at 0 so this will be a Maclaurin series. Remember that approximations get worse the farther we get from the center. (**Refer back to page 491 figure 10.4)**

So, if we can get our polynomial to be close at π (let $x=\pi$, center is "0"), then it will be at least that close on [-π, π].
Now the question is how close and how many terms do we add to get that close? Let's say within .0001 $\left(1\times10^{-4}\right)$

We know the exact value of sin $\pi = 0$ so we have something to compare. The Maclaurin series for sin x is.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \dots$$

Let's put the polynomial into y_1 and then evaluate at π.

at 5 terms $P(\pi)$ .0069 at 7 terms $P(\pi)$ .00002143 6 terms ≈ .00045

(5 terms, then 7, evaluate at $\frac{\pi}{2}$ also) do I need the same amount of terms

.0001

So we say the truncation error is less than .0001 $\boxed{\text{No}}$

closer to center needs fewer terms

We can also look at the error graphically.

(This is why we put the polynomial in y_1) Graph $|P_{13}(x) - \sin x|$, in the window [$-\pi$, π] by [-.00004,.00004]

change window to see error.

Taylor polynomials give us a good way to get an approximation for the function, but how do we know how far off we are without having the exact function value to compare it with.

Example 2: Find a formula for the truncation error if we use $1 + x^2 + x^4 + x^6$ *to approximate* $\frac{1}{(1-x^2)}$ over the interval (-1, 1)

$$\frac{1}{1-x^2} = \underbrace{1 + x^2 + x^4 + x^6}_{\text{4th partial sum for } \frac{1}{1-x^2}} \Big| \underbrace{+ x^8 + x^{10} + x^{12} + \dots + x^{2n} + \dots}_{\text{what is missing}}$$

$a = 1$
$r = x^2$

what is missing
what was thrown away
Error
Remainder

error: $x^8 + x^{10} + x^{12} + \dots + x^{2n} + \dots$

sum of the remaining terms

$$\frac{a}{1-r} \quad \boxed{\frac{x^8}{1-x^2}}$$

On infinite geometric series the remainder is the sum of the remaining terms.

Let's look at a simple example where we can figure out the approximation and know the exact error.

Find a 4th order polynomial to approximate $f(x) = \frac{1}{1+x}$ and find a formula for the error.

$a = 1$
$r = -x$

function power series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \Big| \underbrace{-x^5 + x^6 \ldots\ldots}$$

$$[approximation\;\;][error\;\;]$$

remainder

abs value of what is thrown away

(infinite geometric sum is $\frac{a}{1-r}$)

The error is geometric with r = -x so if we combine the terms from $-x^5$ *on we get* $E = \frac{-x^5}{1--x} = \frac{-x^5}{1+x}$

If we increased the order of our polynomial we would decrease our error.

$$P_n \rightarrow E = \frac{(-x)^n}{1+x}$$

means more terms gives less error.

We can graphically see the error decreasing by making

$y_1 = P_4$ *and* $y_2 = \frac{1}{1+x}$ or since we have the formula

$$y_1 = \frac{(-x)^5}{1+x} \Leftarrow$$

formula of function for this $P_4(x)$ error

The previous problem was fairly simple, especially where finding the error was concerned. The error was the sum of an infinite geometric series, so we could say exactly what the error was. We would like to be able to do this even if the remainder is not geometric. Remember this section is about using polynomials to make approximations and about knowing what the error is. Luckily, Taylor has figured out both of these things for us.

Theorem 3: Taylor's theorem with Remainder

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$\boxed{f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n} \ldots + \underbrace{R_n(x)}_{\text{remainder}}$$

$$where\ \boxed{R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}$$

For some c between a and x.

center x value

Taylor's formula to n^{th} term or order n.

The first equation in Taylor's Theorem is **Taylor's formula**.

The function $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ is the **remainder of order n** or the **error term** for the approximation of f by $P_n(x)$ over I. This is called the **Lagrange form** of the remainder and bounds on $R_n(x)$ found using this form are **Lagrange error bounds**.

With $R_n(x)$ gives us a mathematically precise way to define what we mean when we say that a Taylor series converges to a function on an interval. If $R_n(x) \to 0$ *as* $n \to \infty$ for all x in I, we say that the Taylor series generated by f at x = *a* **converges to f** on I, and we write $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Read: Bounding the Remainder p 502

If we can find a number that this error can't exceed, we call it a Lagrange error bound. This leads to another more useful error theorem.

Proving a function converges to a series.

as $n \to \infty$ a factorial grows faster than an exponential function.

Theorem 4: Remainder Estimation Theorem

If there are positive constants M and r such that $\left|f^{(n+1)}(t)\right| \le \boxed{Mr^{n+1}}$ ← maximum number $f^{n+1}(c)$ could be

n+1 derivative of function

for all t between a and x, then the remainder $R_n(x)$ in Taylor's Theorem satisfies the inequality

→ max of n+1 derivative at "c"

$$\left|R_n(x)\right| \le M \frac{r^{n+1}\left|x-a\right|^{n+1}}{(n+1)!} \qquad \left|R_n(x)\right| \le \frac{f^{n+1}(c)(x-a)^{n+1}}{(n+1)!}$$

If these conditions hold for every n and all the other conditions of Taylor's Theorem are satisfied by f, then the series converges to f(x).

In the error formula, the only real mystery is what the next derivative is and what the n+1 derivative is evaluated at some point between x and a. If we can find a bound for that, we've found a bound for the error…a Lagrange error bound.

Let's use these ideas to do a simple problem and then one that's not so simple.

Prove that $\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$ converges to cos(x) for all x.

$1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots R_n(x)$ power series for cos(x)

For this to converge the remainder must always go to zero as the order of our polynomial goes to infinity. $n\to\infty$ The remainder for the nth order is: $R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ $R_n(x)\le\frac{f^{n+1}(c)\,x^{n+1}}{(n+1)!}$

This approximation is centered at 0 so a = 0. This means we have to pick a point between 0 and x to evaluate the n+1 derivative. This seems awful, but we can say something unique about the n+1 derivative. The most it can be is 1. So….

Range cosx is [-1,1]

The error can't be more than $\frac{1(x-0)^{n+1}}{(n+1)!}$ In notation

$|R_n(x)|\le\frac{1|x^{n+1}|}{(n+1)!}$ Then $|R_n(x)|\le\frac{1|x^{n+1}|}{(n+1)!}\to 0$ because $\to 0$ $\le$ goes to function

Sometimes the derivative doesn't work out as well so we have to use the formula to come up with a Lagrange error bound.

Example 4

$$e^{7x} = \sum_{n=0}^{\infty} \frac{(7x)^n}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

replaced x with 7x

$$\sum_{n=0}^{\infty} \frac{(7x)^n}{n!} \Rightarrow 1 + 7x + \frac{(7x)^2}{2!} + \frac{(7x)^3}{3!} + \dots + \frac{(7x)^n}{n!}$$

$$R_n(x) \le \frac{f^{n+1}(c)(x-0)^{n+1}}{(n+1)!}$$

$f(x) = e^{7x}$

$f'(x) = 7e^{7x}$

$f''(x) = 7^2 e^{7x}$

$f'''(x) = 7^3 e^{7x}$

$f^n(x) = 7^n e^{7x}$

$f^{n+1}(x) = 7^{n+1} e^{7x}$

$$R_n(x) \le \frac{(7^{n+1} e^{7x})\, x^{n+1}}{(n+1)!}$$

as $n \to \infty$ $\to 0$

Satisfies rules so proven that $\sum_{n=0}^{\infty} \frac{(7x)^n}{n!}$ is e^{7x}

Example 5

$$\ln(1+x) \approx x - \frac{x^2}{2} \qquad |x| \le .1$$

$$|R_2(x)| \le \frac{f^3(c)(x)^3}{3!} \qquad -.1 \le x \le .1$$

$f(x) = \ln(1+x)$

$f'(x) = \frac{1}{1+x}$

$f''(x) = \frac{-1}{(1+x)^2}$

$f'''(x) = \frac{2}{(1+x)^3}$

$\frac{2}{(1+.1)^3} \approx 1.50262$

$\frac{2}{(1+-.1)^3} \approx 2.74348$

$$|R_2(x)| \le \frac{\frac{2}{(1+-.1)^3}(-.1)^3}{3!}$$

$\le |-4.57247 \times 10^{-4}|$

$\le 4.572 \times 10^{-4}$

$.0004572$

33. A cubic Approximation of e^x *The approximation* $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ is used on small intervals about the origin. Estimate the magnitude of the approximation error for $|x| \le 0.1$ $-.1 \le x \le .1$

$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ 3rd order

$|R_3(x)| \le \frac{\max (x-0)^4}{4!}$

$f(x) = e^x$
$f'(x) = e^x$
$f''(x) = e^x$
$f'''(x) = e^x$
$f^4(x) = e^x$

$|R_3(x)| \le \frac{f^4(c)(x-0)^4}{4!}$

$|R_3(x)| \le \frac{e^{.1}(.1)^4}{4!}$

use maximum
$f^4(.1) = e^{.1} \approx 1.10517$
$f^4(-.1) = e^{-.1} \approx .9048$

4.60487×10^{-6}
$.0000046$
L. Error Bound

Graphically $\left|e^x - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\right| \approx 4.251 \times 10^{-6}$
True error

20. If cos x is replaced by $1 - \left(\frac{x^2}{2}\right)$ *and* $|x| < 0.5$, what estimate can be made of the error? Does $1 - \left(\frac{x^2}{2}\right)$ tend to be too large or too small? Support your answer graphically.

$\cos x \approx 1 - \left(\frac{x^2}{2}\right)$ 2nd order

$|x| < .5$
$-.5 < x < .5$

$P_2(x) = 1 - \frac{x^2}{2}$
$P_3(x) = 1 - \frac{x^2}{2}$ 3rd order is same as 2nd order
so must use n=3 on Remainder!

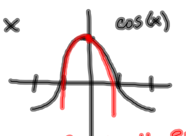
$|R_3(x)| \le \frac{f^4(c)(x-0)^4}{4!}$ max of $\frac{d}{dx}\cos x$ is 1.

$\le \frac{1(x)^4}{4!}$

$\le \frac{1(.5)^4}{4!}$

$\le .00260$

P_2 is too small or below cos x
quadratic opens down



Graphically Error
is .002583
or .877583
$-$ .875
.002583
at x=.5

Gives us error so we need x interval

#19.

Remainder is $\leq 5 \times 10^{-4}$ so what is $|x| \leq$?

$\sin x \approx x - \frac{x^3}{6}$ 3rd order, but 4th order is same so....

use $R_4(x)$

max of $\frac{d}{dx} \sin x = 1$

$$|R_4(x)| \leq \frac{\max f^5 \, x^5}{5!}$$

$$|R_4(x)| \leq \frac{1(x^5)}{5!} \leq 5 \times 10^{-4}$$

$$\frac{x^5}{5!} \leq .0005$$

SOLVE for x

$$x^5 \leq .0005(5!)$$

$$x^5 \leq .0005(120)$$

$$x^5 \leq .06$$

$$\sqrt[5]{x} \leq \sqrt[5]{.06}$$

$$x \leq .5697$$

$$-.56 < x < .56$$

$$\boxed{|x| < .56}$$