Taylor Series

The things we have learned about power series and geometric series so far are nice, but they are only useful for polynomials that fit the geometric form and we didn't use a great deal of calculus. In this section we will extend the ideas we learned in 9.1 and make more general conclusions about series and use more of our calculus skills.

Exploration 1:

Make a polynomial $P(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$ that meets the following conditions:

$P(0)=5, P'(0)=7, P''(0)=11, P'''(0)=13, P^{(4)}(0)=17$

$P(x)=a_0+a_1x^1+a_2x^2+a_3x^3+a_4x^4$ $P(0)=5$ $\boxed{a_0=5}$

$P'(x)=a_1+2a_2x+3a_3x^2+4a_4x^3$ $P'(0)=7$ $\boxed{a_1=7}$

$P''(x)=2a_2+6a_3x+12a_4x^2$ $P''(0)=11$ $2a_2=11$, $a_2=\frac{11}{2}$

$P'''(x)=6a_3+24a_4x$ $P'''(0)=13$ $6a_3=13$, $a_3=\frac{13}{6}$

$P^4(x)=24a_4$ $P^4(0)=17$ $24a_4=17$, $a_4=\frac{17}{24}$

$P(x)=5+7x+\frac{11}{2}x^2+\frac{13}{6}x^3+\frac{17}{24}x^4$

$P(x)=P(0)+P'(0)x+\frac{P''(0)x^2}{2}+\frac{P'''(0)x^3}{6}+\frac{P^4x^4}{24}$

$$\boxed{P(x)=\frac{P(0)}{0!}+\frac{P'(0)x^1}{1!}+\frac{P''(0)x^2}{2!}+\frac{P'''(0)x^3}{3!}+\frac{P^4x^4}{4!}}$$

This is a long tedious process, but things happen in a systematic way so a pattern develops. The idea is to use a technique like this to model functions near zero. As you might suspect, once we center them at zero, we can then center them somewhere else to model their behavior there.

There are several functions of primary interest.

ln, e^x, $\cos x$, $\sin x$

Example 1 $P(x)=a_0+a_1x^1+a_2x^2+a_3x^3+a_4x^4$

$P(x)=f(0)+f'(0)x+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\frac{f^4(0)x^4}{4!}$

$\ln(1+x)$

$f(0)$ $\ln(1+x)\big|_{x=0}$ $\ln 1=0$

$f'(x)$ $\frac{1}{1+x}\big|_{x=0}$ 1

$f''(x)$ $\frac{-1}{(1+x)^2}\big|_{x=0}$ -1

$f'''(x)$ $\frac{2}{(1+x)^3}\big|_{x=0}$ 2

$f^4(x)$ $\frac{-6}{(1+x)^4}\big|_{x=0}$ -6

$P_4(x)=0+1x^1+\frac{-1x^2}{2!}+\frac{2x^3}{3!}+\frac{-6x^4}{4!}$ ($\cancel{3\cdot2\cdot1}$, $\cancel{4\cdot3\cdot2\cdot1}$)

$$\boxed{P_4(x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}}$$

n^{th} term $\frac{(-1)^{n-1}x^n}{n}$

$n=1$

This type of polynomial has a special name; it's called a Taylor Polynomial.
A Taylor Polynomial is a polynomial constructed using derivatives and initial conditions of derivatives to approximate a function near zero. A Taylor Series is an infinite Taylor polynomial that converges to the function near zero. have n^{th} term

If we use only up to the second derivative to construct the polynomial it's called a 2^{nd} order Taylor Polynomial. If we continue to the 5^{th} derivative, it's called a 5^{th} order, etc. Of course, the higher the order found, the better the approximation. A Taylor Series would be of infinite order so it would equal the function on its radius of convergence. It would cease to be an approximation on that interval.

Here is another essential function.
Example 2: Construct the seventh order Taylor polynomial and the Taylor series for sin(x) at x = 0

$f(x)$ $\sin x\big|_{x=0}$ 0 $\qquad$ $f^4(x)$ $\sin x\big|_{x=0}$ 0

$f'(x)$ $\cos x\big|_{x=0}$ 1 $\qquad$ $f^5(x)$ $\cos x\big|_{x=0}$ 1

$f''(x)$ $-\sin x\big|_{x=0}$ 0 $\qquad$ $f^6(x)$ $-\sin x\big|_{x=0}$ 0

$f'''(x)$ $-\cos x\big|_{x=0}$ -1 $\qquad$ $f^7(6)$ $-\cos x\big|_{x=0}$ -1

$$P_7(x) = f(0)+f'(0)x+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\frac{f^4(0)x^4}{4!}+\frac{f^5(0)x^5}{5!}+\frac{f^6(0)x^6}{6!}+\frac{f^7(0)x^7}{7!}$$

$$P_7(x) = 0+1x+\frac{0x^2}{2!}+\frac{-1x^3}{3!}+0+\frac{1x^5}{5!}+0+\frac{-1x^7}{7!}$$

$$P_7(x) = 1x^1-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!} \qquad 7^{th} \text{ order}$$

series
$$P(x) = \underset{n=0}{x} \underset{n=1}{-\frac{x^3}{3!}}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\dots$$

Remember once you have the initial condition for the nth derivative, the coefficient for x^n *is* $\boxed{\dfrac{f^{(n)}(0)}{n!}}$

Notice how the initial conditions appear in a systematic way, even though this is a trig function. This is easier than ln.

Look at figure 10.4.

Instead of our polynomials, our Power Series, our Taylor Series only giving us the correct function values on an interval of convergence, these series will give us <u>the correct answer</u> no matter what x we plug in.

Interval of converge for sin(x) is $\mathbb{R}$

Definition:
Taylor Series Generated by f at x = 0 (Maclaurin Series)
Let *f* be a function with derivatives of all orders throughout some open interval containing 0. Then the **Taylor series generated by f at x = 0** is

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

This series is also called the **<u>Maclaurin series generated by f.</u>**

The partial sum $P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k$ is the **Taylor polynomial of order n for f at x = 0.**

Find the sixth order Taylor polynomial that approximates y = cos (x) at x=0. Find the series for cos (x).

$f(x)\ \cos(x)\big|_{x=0}\ 1$ $\quad f^{4}(x)\ \cos x\big|_{x=0}\ 1$

$f'(x)\ -\sin x\big|_{x=0}\ 0$ $\quad f^{5}(x)\ -\sin x\big|_{x=0}\ 0$

$f''(x)\ -\cos x\big|_{x=0}\ -1$ $\quad f^{6}(x)\ -\cos x\big|_{x=0}\ -1$

$f'''(x)\ \sin x\big|_{x=0}\ 0$

$$P_6(x)=1+0x-\frac{x^2}{2!}+\frac{0x^3}{3!}+\frac{1x^4}{4!}+\frac{0x^5}{5!}+\frac{-1x^6}{6!}$$

$$P_6(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}$$

$$\underset{\text{series}}{P(x)}=\underset{n=0}{1}-\underset{n=1}{\frac{x^2}{2!}}+\underset{n=2}{\frac{x^4}{4!}}-\underset{n=3}{\frac{x^6}{6!}}+\dots+\frac{(-1)^n x^{2n}}{(2n)!}+\dots$$

Example 3: Find the fourth order Taylor polynomial that approximates y = cos (2x) near x = 0.

$f(x)\ \cos(2x)\big|_{x=0}\ 1$

$f'(x)\ -2\sin(2x)\big|_{x=0}\ 0$

$f''(x)\ -4\cos(2x)\big|_{x=0}\ -4$

$f'''(x)\ +8\sin(2x)\big|_{x=0}\ 0$

$f^{4}(x)\ 16\cos(2x)\big|_{x=0}\ 16$

$$P_4(x)=1-\frac{4x^2}{2!}+\frac{16x^4}{4!}$$

We know $\cos(x)$

$$P_4(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}$$

we replace x with $2x$

$$\cos(2x)\Rightarrow P_4(x)=1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}$$

$$1-\frac{4x^2}{2!}+\frac{16x^4}{4!}$$

5. Find the first 3 terms of the Maclaurin series for sin (2x).

$$\sin(x)\approx x^1-\frac{x^3}{3!}+\frac{x^5}{5!}$$

$$\sin(2x)\approx(2x)^1-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}+\dots+\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}$$

$$2x-\frac{8x^3}{3\cdot2\cdot1}+\frac{32x^5}{5\cdot4\cdot3\cdot2\cdot1}$$

BOB

$$2x-\frac{4x^3}{3}+\frac{4}{15}x^5+\dots+\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}+\dots$$

Interval of convergence is $\mathbb{R}$

As we said at the beginning of the section, once we have a series that represents a function near zero, we would want to make the series so it approximates the function elsewhere.

Definition: Taylor Series Generated by f at x = a
Let f be a function with derivatives of all orders throughout some open interval containing a. Then the **Taylor series generated by f at x = a** is centered around "a" not 0

$$f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

The partial sum $P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$ is the **Taylor polynomial of order n for f at x = a.**

You will notice that where there used to be a zero, there is now an "a". That is essentially the only difference.

Example 4:
Find the Taylor series generated by f(x) = e^x at x = 2.

$f(x)=e^x$ at $x=2$

$f(x) \quad e^x\big|_{x=2} \quad e^2$

$f'(x) \quad e^x\big|_{x=2} \quad e^2$

$f''(x) \quad e^x\big|_{x=2} \quad e^2$

$$f(2)+f'(2)(x-2)^1+\frac{f''(2)(x-2)^2}{2!}+\ldots+\frac{f^n(2)(x-2)^n}{n!}+\ldots$$

$$e^2+e^2(x-2)^1+\frac{e^2(x-2)^2}{2!}+\ldots+\frac{e^2(x-2)^n}{n!}+\ldots$$

$$\sum_{k=0}^{\infty}\left(\frac{e^2}{k!}\right)(x-2)^k$$

Example:
Let $f(x)=\ln x$ Find the Taylor series generated by f(x) at x = 2.

$f(x) \quad \ln x\big|_{x=2} \quad \ln 2$

$f'(x) \quad \frac{1}{x}\big|_{x=2} \quad \frac{1}{2}$

$f''(x) \quad \frac{-1}{x^2}\big|_{x=2} \quad \frac{-1}{4}$

$f'''(x) \quad \frac{2}{x^3}\big|_{x=2} \quad \frac{2}{8}$

$f^4(x) \quad \frac{-6}{x^4}\big|_{x=2} \quad \frac{-6}{16}$

$$\underset{n=0}{\ln 2}+\underset{n=1}{\frac{1}{2}(x-2)}+\frac{-1}{4}\frac{(x-2)^2}{2!}+\frac{2}{8}\frac{(x-2)^3}{3!}+\frac{-6(x-2)^4}{16\cdot 4!}+\ldots$$

$$+\frac{(-1)^{n+1}(n-1)!(x-2)^n}{2^n\ n!} \qquad \frac{(n-1)!}{n!}=\frac{\cancel{(n-1)!}}{n\cancel{(n-1)!}}$$

n^{th} term

$$+\frac{(-1)^{n+1}(x-2)^n}{2^n\cdot n}$$

$$\ln 2+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(x-2)^k}{2^k\cdot k}$$

Example 5:

$f(x) = 2x^3 - 3x^2 + 4x - 5$

at $x=0$

$f(0) = -5$

$f'(x) = 6x^2 - 6x + 4 \big|_{x=0} \quad 4$

$f''(x) = 12x - 6 \big|_{x=0} \quad -6$

$f'''(x) = 12 \big|_{x=0} \quad 12$

$P_3(x) = -5 + 4x + \frac{-6x^2}{2!} + \frac{12x^3}{3!}$

$\boxed{-5 + 4x - 3x^2 + 2x^3}$

at $x=1$ center $a=1$

$f(x) \quad 2x^3 - 3x^2 + 4x - 5 \big|_{x=1} \quad -2$

$f'(x) \quad 6x^2 - 6x + 4 \big|_{x=1} \quad 4$

$f''(x) \quad 12x - 6 \big|_{x=1} \quad 6$

$f'''(x) \quad 12 \big|_{x=1} \quad 12$

$-2 + 4(x-1) + \frac{6(x-1)^2}{2!} + \frac{12(x-1)^3}{3!}$

$\boxed{-2 + 4(x-1) + 3(x-1)^2 + 2(x-1)^3}$

$-2 + 4x - 4 + 3(x-1)^2 + 2(x-1)^3$

$-6 + 4x + 3(x-1)^2 + 2(x-1)^3$

Combining Taylor Series

This can get very complicated, so luckily they help us out with some nice tools. First, as long as you are working on the interval of convergence, Taylor series can be added, subtracted, and multiplied by other polynomials. Second, they give us a table of Maclaurin series. (Page 495) (Memorize these!)

So if we wanted a Taylor series for $\frac{x}{1+x}$ we take the Maclaurin series $\frac{1}{1+x}$ and multiply by x. If we want the series for $\frac{xe^x}{2}$, we take the series, multiply by x and divide by 2.

Remember order of operations

$\frac{x\cos(3x)}{5}$

write $\cos(x)$

replace x with $(3x)$

mult by x

$\div 5$