10.1 notes Calculus

Infinite Series

Power Series:

A sequence is simply a list of numbers. They can be finite or infinite.

Examples: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ $\qquad$ $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\ldots$

A series is a sequence of numbers that is added together. They can also be finite or infinite.

Example: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ $\qquad$ $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots$

Any finite series would have a sum, but the same is not true for an infinite series. Sometimes it will have a sum, or **converge** to a value, other times it will not converge, it will **diverge**.

It might seem strange to study such things, but remember we started our study of calculus by looking at limits: going to infinity, getting infinitely close and we've added up an infinite number of infinitely thin rectangles to get an integral. This is just the next step.

Definition: Infinite Series
An infinite series is an expression of the form

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots \ \ or \ \ \sum_{k=1}^{\infty} a_k$$

The numbers $a_1, a_2, \cdots$ are the **terms** of the series; a_n is the **nth term**.

Example: $\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots$

Never forget that all the normal rules for addition still apply: (associative and commutative)

If we decide to stop at any part, we get what is called a "**Partial Sum**".

Example:

$$\sum_{k=1}^{2} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \qquad S_2 = \frac{3}{2}$$

$$\sum_{k=1}^{3} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \qquad S_3 = \frac{11}{6}$$

This is called a sequence of partial sums.

If the sequence of partial sums has a limit as n→∞ then we say the series **converges**. In other words, the series adds up to a finite number. . In other words, we have to get to a point where we are adding essentially a zero. But that isn't the only thing that determines convergence.

If the series does not converge, then we say it diverges.

Diverges Converges

Examples: $\sum_{n=1}^{\infty} \frac{1}{2n}$ $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\ldots$ $\sum_{n=1}^{\infty} \frac{1}{2^n}$ $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$

Both series have terms that tend to zero, but the partial sums of the first continue to increase so the series diverges. The partial sums of the second have a finite limit so the series converges.

Here are some examples of divergent series.

$$\sum_{k=1}^{\infty} k \qquad \sum_{n=1}^{\infty} n^2 \qquad \sum_{n=1}^{\infty} (-1)^n$$

$1+2+3+4+\dots$

diverges

$1^2+2^2+3^2+\dots$

diverges

$(-1)^1+(-1)^2+(-1)^3+(-1)^4+\dots$

$\underbrace{-1+1}+-1+1+\dots$

is it 0 or is it -1

What about this series?

$$\sum_{n=1}^{\infty} \frac{6}{10^n} \qquad \sum_{n=1}^{\infty} 6\left(\frac{1}{10}\right)^n$$

$$\frac{6}{10}+\frac{6}{10^2}+\frac{6}{10^3}+\frac{6}{10^4}+\dots$$

$$\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}+\frac{6}{10000}+\dots$$

$.6+.06+.006+.0006+\dots$

$.6666$

$.\overline{6}$

$\boxed{\text{converge } \frac{2}{3}}$

geometric $\frac{a}{1-r}$

$$\frac{\frac{6}{10}}{1-\frac{1}{10}}$$

$$\frac{6}{10} \div \frac{9}{10}$$

$$\frac{6}{\cancel{10}} \cdot \frac{\cancel{10}}{9} \qquad \frac{6}{9} \qquad \frac{2}{3}$$

If the **infinite series** $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots + a_k + \cdots$ converges, then $\lim_{n\to\infty} a_k = 0$

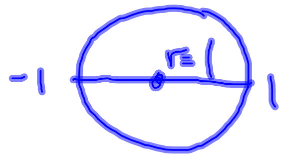
This means that if $\lim_{k\to\infty} a_k \neq 0$ this series must diverge.

$a_1 + ar + a \cdot r \cdot r + a \cdot r \cdot r \cdot r$

The **geometric series** $a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = \sum_{n=1}^{\infty} ar^{n-1}$

MULT by r each time

converges to the $\boxed{\text{sum } \frac{a}{1-r}}$ $\boxed{\text{if } |r|<1}$ *and diverges if* $|r| \geq 1$. The interval of convergence is (-1, 1) or $-1 < r < 1$



Example 3

a) $\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1}$

$a_1 = 3$, $r = \frac{1}{2}$

$a_2 = 3\left(\frac{1}{2}\right)^1$ $|r| < 1$

$a_3 = 3\left(\frac{1}{2}\right)^2$ $\left|\frac{1}{2}\right| < 1$

sum $\frac{a}{1-r} \Rightarrow \frac{3}{1-\frac{1}{2}}$ $\frac{3}{\frac{1}{2}}$ $3 \div \frac{1}{2}$ $3 \cdot \frac{2}{1} = \boxed{6}$

converge

b) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + \left(-\frac{1}{2}\right)^{n-1} + \ldots$

$a_1 = 1$ $r = -\frac{1}{2}$ $\left|-\frac{1}{2}\right| < 1$ yes

$\frac{a}{1-r}$ $\frac{1}{1-\frac{-1}{2}} = \frac{1}{\frac{3}{2}} \Rightarrow \boxed{\frac{2}{3}}$ converge

c) $\sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k$ $a_0 = 1$, $r = \frac{3}{5}$ $\left|\frac{3}{5}\right| < 1$ yes $\frac{1}{1-\frac{3}{5}}$

converges $\frac{1}{\frac{2}{5}} = \boxed{\frac{5}{2}}$

d) $\frac{\pi}{2} + \frac{\pi^2}{4} + \frac{\pi^3}{8} + \ldots$

MULT by $\frac{\pi}{2}$

$\left|\frac{\pi}{2}\right| < 1$ No $\therefore$ $\boxed{\text{diverges}}$

Representing Functions by Series

If $|x| < 1$, then the geometric series formula assures us that

MULT by x each time Sum $\frac{a}{1-r}$

$1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{1}{1-x}$. The partial sums of the infinite series on the left are all polynomials, so we can graph them.

Look at **Figure 10.1**

The expression $\sum_{n=0}^{\infty} c_n x^n$ is like a polynomial in that it is a sum of coefficients times powers of x, but polynomials have finite degrees and do not suffer from divergence for the wrong values of x. Just as an infinite series of numbers is not a mere sum, this series of powers of x is not a mere polynomial.

Definition: Power Series

An expression of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n + \ldots$$ is a **power series**

NOTHING IS subtrated from x

centered at x = 0. An expression of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \ldots + c_n (x-a)^n + \ldots$$ is

a **power series centered at $\boxed{x = a}$.** The term $c_n (x-a)^n$ is the **nth term;** the number a is the **center.**

The power series $\sum_{n=0}^{\infty} x^n$ represents the function $\frac{1}{(1-x)}$ on the domain (-1,1). We can now take our basic series $\sum c_n x^n = \frac{c_n}{1-x}$ and move it so it's interval of convergence is no longer centered at zero.

Can we come up with other power series for other functions? $\frac{a}{1-r}$

Example: $\frac{2}{1-x}$ $= \underset{n=1}{2} + 2x + 2x^2 + 2x^3 + \cdots + 2x^{n-1} + \cdots$

$a = 2$

$r = x$

$\underset{n=0}{2} + 2x + 2x^2 + 2x^3 + \cdots + 2x^n + \cdots$

Example: $\frac{1}{1+x}$ $= \underset{n=0}{1} - x + x^2 - x^3 + x^4 - \cdots (-x)^n + \cdots$

$a = 1$

$r = -x$

$n=1$

$(-1)^n x^n$

$(-x)^{n-1}$

$(-1)^{n-1} x^{n-1}$

Example: $\frac{x}{1+x}$ $= \underset{n=1}{x} - x^2 + x^3 - x^4 + \cdots + (-1)^{n-1} x^n + \cdots$

$a = x$

$r = -x$

$\underset{n=0}{x} - x^2 + x^3 - x^4 + \cdots + (-1)^n x^{n+1} + \cdots$

$+ x(-x)^n + \cdots$

Example: $\dfrac{1}{1-2x} = \underset{n=0}{1} + 2x + 4x^2 + 8x^3 + \dots + (2x)^n + \dots$

$a = 1$

$1 - 2x = 1 - r$

$-2x = -r$

$2x = r$

Example: $\dfrac{1}{x} = \dfrac{1}{1+(x-1)}$

$a = 1$

$1 - r = x$

$-r = x - 1$

$r = -x + 1$

$r = -(x-1)$

$= 1 - (x-1)^1 + (x-1)^2 - (x-1)^3 + (x-1)^4 + \dots (-1)^n (x-1)^n + \dots$

21. Find the interval of convergence and the function of x represented by the geometric series $\sum_{n=0}^{\infty} 2^n x^n$

$\sum_{n=0}^{\infty} 2^n x^n$

$a_0 = 2^0 x^0 = 1$

$a_1 = 2^1 x^1 = 2x$

$a_2 = 2^2 x^2 = 4x^2$

$f(x) = \frac{a}{1-r}$

$\boxed{f(x) = \frac{1}{1-2x}}$

$\sum_{n=0}^{\infty} (2x)^n$

$a_0 = 1 \quad r = 2x$

$|r| < 1$

$|2x| < 1$

$|x| < \frac{1}{2}$ radius of convergence

$-1 < 2x < 1$

$-\frac{1}{2} < x < \frac{1}{2}$

interval of convergence

$\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$

We have covered one rather cumbersome way to use a power series to represent other functions. This was all based on adjustments of the geometric series. We have been using tools all year that give us functions from other functions. For instance, we can get 2x from x^2. We can get sin(x) from –cos(x) and we can get e^x from e^x. The tools we use are integration and differentiation. We should be able to use the same tools to get new functions from power series.

Example 4:

Given that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$. On the

$a = 1 \; ; \; r = x$

interval (-1,1), find a power series to represent $\frac{1}{(1-x)^2}$

$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}$ term by term $\quad \frac{d}{dx}\left((1-x)^{-1}\right) = -1(1-x)^{-2}(-1)$

$(1-x)^{-2}$

$\frac{1}{(1-x)^2}$

$\frac{d}{dx}\left(1 + x + x^2 + x^3 + \ldots + x^n + \ldots\right)$

$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots + nx^{n-1} + \ldots$

We can do the same by integrating

Example 5

$\frac{1}{1+x}$ Find the power series to represent $\ln(1+x)$

$\int \frac{1}{1+x}\,dx \qquad \int 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots \, dx$

$a=1 \quad r=-x$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^n x^{n+1}}{(n+1)} + \dots}$$

Remember: The formula tells us that anything in the form

$1+x+x^2+x^3+\ldots+x^n+\ldots=\dfrac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ as long as

$|x|<1$ $\quad -1<x<1$ or more generally $\dfrac{a}{1-r}$, $|r|<1$ $\sum_{n=0}^{\infty}ar^n$

Change r to x $=\dfrac{a}{1-x}$

This tells us that a series is capable of representing a function....at least on its interval of convergence.

A series can represent a function on its interval of convergence. This is similar to the way a tangent line approximates a graph at the point of tangency or linearizing a function.
Let's look at a special type of series and the family of functions they represent.

Exploration 2

$\frac{1}{1+x^2}$ $(-1,1)$ interval of convergence

$\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx$ $a=1$, $r=-x^2$

① $\int 1 - \underset{n=1}{x^2} + \underset{n=2}{x^4} - \underset{n=3}{x^6} + \cdots + (-1)^n x^{2n} + \cdots\, dx$

② $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$

③ $y_1 = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$

$y_2 = \tan^{-1}(x)$

radian mode

$(-1,1)$ $\boxed{\text{yes}}$

④ when $x=1$ $y_1(1) = .7238$ $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$

$y_2(1) = .78539$

$\tan^{-1}(1) = \frac{\pi}{4}$ $\frac{\pi}{4} \approx .7853$

converges at $x=1$

actually the interval of converges $[-1,1]$

$|x| \le 1$

Exploration 3

$$f(x) = \underset{n=0}{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots$$

① $f'(x) = 0 + 1 + \frac{\cancel{2}x}{\cancel{2}\cdot 1} + \frac{\cancel{3}x^2}{\cancel{3}\cdot 2\cdot 1} + \frac{\cancel{4}x^3}{\cancel{4}\cdot 3\cdot 2\cdot 1} + \dots + \frac{nx^{n-1}}{n!} + \dots$

$$f'(x) = \underset{n=1}{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{\cancel{n}\, x^{n-1}}{\cancel{n}\,(n-1)!}$$

② $f(0) = 1$ if $x=0$ $f(0)=$

initial condition

③ e^x

④ $y = f(x) \Rightarrow f'(x)$

$\frac{dy}{dx} = f'(x)$

$\frac{dy}{dx} = y$

$\int \frac{dy}{y} = \int dx$

$\ln|y| = x + C$

$\ln y = x + C$

$\ln 1 = 0 + C$

$0 = C$

$\ln y = x$

$\log_e y = x$

$\boxed{y = e^x}$

⑥ $y_1 = 1 + x + \frac{x^2}{2}$

$y_2 = e^x$

$(-1, 1)$

⑦ $y_1 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$

$y_2 = e^x$

Interval of convergence

$\mathbb{R}$