9.4 notes calculus

Improper Integrals

The name is awful, but as it turns out most of the time there is nothing wrong with the integrals we call improper. The following is an improper integral.

Example: $\int_0^\infty e^{-x}\,dx$

There wouldn't be a problem and nothing would be new if it were: $\int_0^4 e^{-x}\,dx$

The problem here is that one of the limits is infinity. We'll handle this by treating this as a definite integral and then taking the limit as the upper limit goes to infinity. Sometimes there will be an answer, and sometimes there won't be an answer. It will basically come down to how fast the graph approaches the x-axis. If it doesn't approach the x-axis then there will be no integral for sure. If it does approach the x-axis it might. We will put some rules to this in a minute.

Example: $\int_0^\infty e^{-x}dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b$

The reason this works is that e^{-x} approaches zero as x→∞ and it approaches zero rapidly enough that area effectively stops accumulating.

Definition: Improper Integrals with Infinite Integration Limits
Integrals with infinite limits of integration are **improper integrals.**

1. If f(x) is continuous on [a, ∞), then $\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)dx$
2. If f(x) is continuous on (- ∞,b], then $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)dx$
3. If f(x) is continuous on (- ∞, ∞), then $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$ where c is any real number

We say an integral **converges** if the preceding process gives a finite answer. If it doesn't, we say it **diverges.** If we have to break the integral into two parts, they both must converge, otherwise the entire integral diverges.

Example 1: $\int_{-\infty}^{\infty} e^x$

Example 2: $\int_1^\infty \frac{dx}{x}$ Graph, then evaluate

Example 3: $\int_0^\infty \frac{2\,dx}{x^2+4x+3}$

Example 4: $\int_1^\infty xe^{-x}\,dx$

Example 5: $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Problem 3. $\int_{-\infty}^{\infty} \frac{2x\ dx}{\left(x^2+1\right)^2}$

The discontinuities we dealt with before were finite discontinuities. We will now look at infinite discontinuities with the same technique used for improper integrals.

Example: $\int_0^1 \frac{dx}{\sqrt{x}}$

Definition: Improper Integrals with Infinite Discontinuities
Integrals of functions that become infinite at a point within the interval of integration are **improper integrals**.

1. If f(x) is continuous on (a, b], then $\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)dx$
2. If f(x) is continuous on [a,b), then $\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)dx$
3. If f(x) is continuous on $[a,c)\cup(c,b]$, then $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Exploration 1:

The integral $\int_1^{\infty} \frac{dx}{x^P}$

Example 6 $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$

Example 7: $\int_1^2 \frac{dx}{x-2}$

25. $\int_0^2 \frac{dx}{1-x^2}$

28. $\int_0^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx$

There are some integrals that our best tools cannot get analytic answers to

Example 8: $\int_1^\infty e^{-x^2} dx$ Does this integral converge?

This function approaches the x-axis very quickly, but there is no closed form (not a nice one) that we can use in the limit. Our calculator can approximate the answer, but the best we can do analytically is to say if the integral exists.

The simplest way to do this is make a comparison with a function we know. Compare e^{-x^2} *with* e^{-x}
Both are exponentials. Both are positive and both approach zero as x→∞. Which one is larger?
Graph e^{-x^2} *and* e^{-x} Area must be less than e^{-1}. Compute using fnint $(e^{-x^2}, x, 1, x)$

Theorem 6: Direct Comparison Test
Let f and g be continuous on [a , ∞), with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\int_a^\infty f(x)\,dx = converges\ if\ \int_a^\infty g(x)dx\ converges$
2. $\int_a^\infty g(x)\,dx = diverges\ if\ \int_a^\infty f(x)dx\ diverges$

33. $\int_\pi^\infty \frac{2+\cos x}{x} dx$

If convergence depends on how fast f(x) →0 then, we can also determine convergence using relative rates of growth. If two functions grow at the same rate, they should both either converge or diverge together.

Theorem 4: Limit Comparison Test
If the positive functions f and g are continuous on [a , ∞) and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L,\ 0 < L < \infty$$

then

$\int_a^\infty f(x)dx$ *and* $\int_a^\infty g(x)dx$ both converge or both diverge.

Example for LCT: $\int_1^\infty \frac{dx}{1+x^2}$

In general, knowing which comparison to use just takes practice. It is possible that both methods will work. You want to find the easiest

22. $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

We can use improper integrals to find some unusual things.

Example 9   Use the arc length to show that the circumference of the circle $x^2 + y^2 = 4$ is 4π.

Example 10   Find the volume of the solid formed by revolving the curve y = xe^{-x} about the x-axis from 0 to ∞.