Partial Fraction Decomposition

Partial Fraction Decomposition:

Consider adding two rational expressions:

$$\frac{3}{x+4}+\frac{2}{x-3}=\frac{3(x-3)+2(x+4)}{(x+4)(x-3)}=\frac{3x-9+2x+8}{(x+4)(x-3)}=\frac{5x-1}{x^2+x-12}$$

In this section, we are going to reverse the procedure starting with the rational expression $\frac{5x-1}{x^2+x-12}$ and writing it as a sum of two simpler fractions $\frac{3}{x+4}$ and $\frac{2}{x-3}$.

This process is called ***partial fraction decomposition***, and the two simper fractions are called ***partial fractions*.**

Case 1: Denominator Has Only Nonrepeated Linear Factors

Given a rational expression with $Q(x)=(x-a_1)(x-a_2)...(x-a_n)$, the partial fraction decomposition of $\frac{P}{Q}$ takes the form $\frac{P(x)}{Q(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+...+\frac{A_n}{x-a_n}$, where the numbers $A_1, A_2, ..., A_n$ are to be determined.

To Solve:

1. Factor the denominator completely.
2. Set fraction equal to the sum of multiple fractions--one with each factor as a denominator. For numerators, use *A*, *B*, *C* etc. **Make sure you have written this step as an equation!**
3. Clear the equation of fractions by multiplying both sides of the equation by $Q(x)$.

Heaviside Method (Named after Oliver Heaviside, an English electrical engineer, mathematician, and physicist)**:**

4. You now have an equation that is true for any value of x. Choose values to plug in for x that will result in *A*, *B*, *C*, etc. being multiplied by zero and cancelling out. Determine as many of the numerators as possible in this way.
5. If you have not yet solved for all of the variables, plug in any other number(s) for x until you have an equation or a system of equations that can be solved for the remaining variables.
6. Substitute the solutions for *A*, *B*, etc. back into the partial fractions.

Coefficients Method:

1-3. Same as above.

4. Equate the coefficients of like powers and write a system of equations.
5. Solve the system for *A*, *B*, etc.
6. Substitute the solutions for *A*, *B*, etc. back into the partial fractions.

Examples: Write the partial fraction decomposition of the following.

a) $\dfrac{x+2}{x^2-7x+12}$

b) $\dfrac{x^2-9x-6}{x^3+x^2-6x}$

Case 2: Denominator Has Repeated Linear Factors

If the denominator has a repeated linear factor, $(x-a)^n$, where $n \geq 2$, then in the partial fraction decomposition, we allow for the terms $\dfrac{A_1}{(x-a)}+\dfrac{A_2}{(x-a)^2}+\cdots+\dfrac{A_n}{(x-a)^n}$, where the numbers A_1, A_2, ... , A_n are to be determined.

Examples: Write the partial fraction decomposition of the following.

a) $\dfrac{2x^2+7x+23}{(x-1)(x+3)^2}$

b) $\dfrac{x+1}{x^2(x-2)^2}$

Case 3: Denominator Has a Nonrepeated Irreducible Quadratic Factor

If the denominator contains a quadratic factor of the form $ax^2 + bx + c$ that can't be factored, then in the partial fraction decomposition, allow for the term $\dfrac{Ax+B}{ax^2+bx+c}$, where the numbers A and B are to be determined.

Examples: Write the partial fraction decomposition of the following.

a) $\dfrac{2x+4}{x^3-1}$

b) $\dfrac{7x^2-25x+6}{\left(x^2-2x-1\right)(3x-2)}$