Matrix Algebra

***Equal Matrices*:** $A = B$ if A and B have the same dimensions and each entry a_{ij} in A is equal to the corresponding entry b_{ij} in B.

***Sums and Differences of Matrices*:** Only matrices of the same dimensions can be added or subtracted. To find a sum or difference, add or subtract corresponding entries of the two matrices.

***Commutative Property of Matrix Addition*:** $A + B = B + A$
***Associative Property of Matrix Addition*:** $(A+B)+C = A+(B+C)$

***Zero Matrix*:** A matrix whose entries are all equal to 0.

***Additive Identity Property of Matrix Addition*:** $A + 0 = 0 + A = A$

Scalar Multiplication*:** If k is a real number and A is a matrix, the matrix kA is the matrix formed by multiplying each entry in A by k. The number k is called a ***scalar*,** and the matrix $\boldsymbol{kA}$ is called a ***scalar multiple of A.

***Properties of Scalar Multiplication*:**

$k(hA) = (kh)A$

$(k+h)A = kA + hA$

$k(A+B) = kA + kB$

Examples: Let $A = \begin{bmatrix} -7 & 2 & 4 \\ 3 & -5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 6 & -8 \\ 0 & 2 & -3 \end{bmatrix}$. Calculate:

a) $A + B$ b) $A - B$ c) $4A$ d) $2A - 3B$

***Row Vector*:** A 1 by n matrix $R = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$

***Column Vector*:** An n by 1 matrix $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

***Product of a Row and a Column*:** $RC = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r_1c_1 + r_2c_2 + \cdots + r_nc_n$

Example: Find $\begin{bmatrix} 5 & 1 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 1 \\ 7 \end{bmatrix}$

Matrix Multiplication: To multiply A and B, the number of columns of A must equal the number of rows of B. The product matrix will have the same number of rows as A and the same number of columns of B.

$$\begin{array}{ccc} A & & B \\ m \text{ by } r & & r \text{ by } n \end{array}$$

Must be the same.

AB is m by n.

The entry in row i, column j of the product matrix is the product of row i of A and column j of B. For example, to find the entry in row 2, column 3 of AB, multiply row 2 of A by column 3 of B.

Examples: Find the following products.

a) $\begin{bmatrix} 2 & 3 & -4 \\ 4 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -5 & 3 \\ 1 & 0 \\ 6 & 2 \end{bmatrix}$

b) $\begin{bmatrix} -5 & 3 \\ 1 & 0 \\ 6 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & -4 \\ 4 & 5 & 1 \end{bmatrix}$

★ **Matrix Multiplication Is Not Commutative!** $AB \neq BA$

***Associative Property of Matrix Multiplication*:** $A(BC) = (AB)C$

***Distributive Property*:** $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$

***Identity Matrix*:** The n by n identity matrix I_n is the n by n matrix with 1's along the main diagonal and 0's everywhere else.

eg) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

***Identity Property*:** If A is an m by n matrix, then $I_m A = A$ and $AI_n = A$.
If A is an n by n square matrix, $AI_n = I_n A = A$.

***Inverse of a Matrix*:** Let A be a square n by n matrix. If there exists an n by n matrix A^{-1}, read "A inverse," for which $AA^{-1} = A^{-1}A = I_n$, then A^{-1} is called the **inverse** of the matrix A.

***Nonsingular Matrix*:** A matrix that has an inverse.

***Singular Matrix*:** A matrix with no inverse.

Finding the Inverse of a Nonsingular Matrix

1. Form the matrix $[A \mid I_n]$.
2. Transform the matrix $[A \mid I_n]$ into reduced row echelon form.
3. The reduced row echelon form of $[A \mid I_n]$ will contain the identity matrix I_n on the left of the vertical bar; the n by n matrix on the right of the vertical bar is the inverse of A.

Examples: Find the inverse of each nonsingular matrix.

a) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Solving a System of Linear Equations Using an Inverse Matrix

The system of equations $\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$ can be rewritten as the matrix equation $AX = B,$ where

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Using matrix algebra, we can show:

$$\begin{aligned} &AX = B \\ &A^{-1}(AX) = A^{-1}B \\ &(A^{-1}A)X = A^{-1}B \\ &I_n X = A^{-1}B \\ &\boldsymbol{X = A^{-1}B} \end{aligned}$$

★ $\boldsymbol{X = A^{-1}B}$ means that to find the solution of the system of equations, multiply the inverse of the coefficient matrix by the column matrix formed from the constants on the right hand side of the equations.

Examples: Solve each system of equations using the inverses found in the previous example.

a) $\begin{cases} 5x - 2y = 3 \\ 3x - y = -4 \end{cases}$

b) $\begin{cases} x + 2z = 6 \\ -x + 2y + 3z = -5 \\ x - y = 6 \end{cases}$