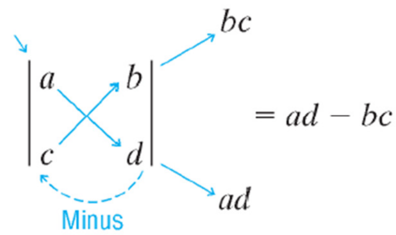
Determinants

If a, b, c, and d are four real numbers, the symbol $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called a ***2 by 2 determinant.*** Its value is the number $ad - bc$; that is,

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$



Examples: Evaluate each 2 by 2 determinant.

a) $\begin{vmatrix} 8 & -3 \\ 4 & 2 \end{vmatrix} =$

b) $\begin{vmatrix} 6 & 1 \\ 5 & 2 \end{vmatrix} =$

c) $\begin{vmatrix} 2 & -3 \\ -7 & -5 \end{vmatrix} =$

d) $\begin{vmatrix} -9 & -7 \\ 1 & -6 \end{vmatrix} =$

3 by 3 Determinants: Expansion by Minors

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The 2 by 2 determinants shown above are called ***minors*** of the 3 by 3 determinant. The ***minor*** $\boldsymbol{M_{ij}}$ is the determinant that results from removing the ith row and jth column of the determinant.

The signs in the equation for a 3 by 3 determinant alternate. To determine whether to add or subtract each term, consider the expression $(-1)^{i+j}$, where $i+j$ is the sum of the row and column number of the entry associated with each minor. If $i+j$ is even, $(-1)^{i+j} = 1$, so we add, and if $i+j$ is odd, $(-1)^{i+j} = -1$, so we subtract.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

+ means "same sign"
– means "opposite sign"

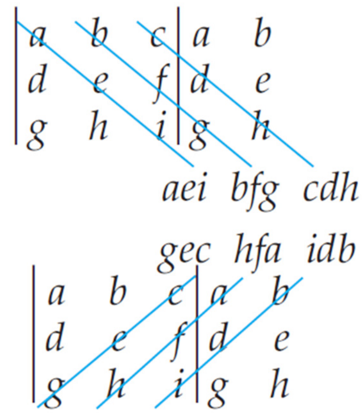
The process used to find the value of a 3 by 3 determinant is called ***expanding across a row or column.*** Select one row or column, and multiply each entry in that row or column by its minor. Use the rules above to decide whether to add or subtract. The value of the determinant is the same no matter which row or column you use.

Examples: Evaluate each 3 by 3 determinant using expansion by minors.

a) $\begin{vmatrix} 1 & 3 & -2 \\ 6 & 1 & -5 \\ 8 & 2 & 3 \end{vmatrix} =$

b) $\begin{vmatrix} -2 & 1 & 0 \\ 1 & 2 & 3 \\ -1 & 4 & 2 \end{vmatrix} =$

3 by 3 Determinants: Using Diagonals



1. Rewrite the first two columns to the right of the determinant.
2. Draw diagonals from each element in the top row downward to the right. Find the product of the entries on each diagonal.
3. Draw diagonals from each element in the bottom row upward to the right. Find the product of the entries on each diagonal.
4. Add the products of the first set of diagonals and subtract the products of the second set of diagonals.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + bfg + cdh) - (gec + hfa + idb)$$

Example: Evaluate each discriminant using diagonals.

a) $\begin{vmatrix} 4 & -1 & 2 \\ 6 & -1 & 0 \\ 1 & -3 & 4 \end{vmatrix}$

b) $\begin{vmatrix} 7 & -1 & 3 \\ -4 & 2 & 2 \\ 0 & 1 & -3 \end{vmatrix}$

Cramer's Rule for a 2 by 2 System of Equations

The solution to the system of equations $\begin{cases} a_{11}x + a_{12}y = c_1 \\ a_{21}x + a_{22}y = c_2 \end{cases}$ or $\left[\begin{array}{cc|c} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \end{array}\right]$ is given by:

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ \text{provided } D \neq 0,\ \text{where}$$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},\ D_x = \begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix},\ \text{and } D_y = \begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}$$

D is the determinant of the coefficient matrix, D_x is formed by replacing the entries in the x-column of D by the constants on the right-hand side of the equation, and D_y is formed by replacing the entries in the y-column of D by the constants on the right-hand side of the equation.

Examples: Solve by using Cramer's Rule.

a) $\begin{cases} 5x - y = 13 \\ 2x + 3y = 12 \end{cases}$

b) $\begin{cases} 2x + 4y = 16 \\ 3x - 5y = -9 \end{cases}$

Cramer's Rule for a 3 by 3 System of Equations

The solution to the system of equations $\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$ or $\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right]$ is given by:

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ z = \frac{D_z}{D},\ \text{provided } D \neq 0,\ \text{where}$$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},\ D_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix},\ D_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix},\ \text{and } D_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}.$$

Examples: Solve by using Cramer's Rule.

a) $\begin{cases} x - y + z = -4 \\ 2x - 3y + 4z = -15 \\ 5x + y - 2z = 12 \end{cases}$

b) $\begin{cases} x + 4y - 3z = -8 \\ 3x - y + 3z = 12 \\ x + y + 6z = 1 \end{cases}$

Properties of Determinants:

1. Interchanging any two rows or columns of a matrix changes the sign of the determinant.
2. If all the entries in any row or column of a matrix equal zero, the value of the determinant is zero.
3. If any two rows or columns of a matrix have corresponding entries that are equal, the value of the determinant is zero.
4. If any row or column of a determinant is multiplied by a nonzero number *k*, the value of the determinant is also changed by a factor of *k*.
5. If the entries of any row or column of a matrix are multiplied by a nonzero number *k* and the result is added to the corresponding entries of another row or column, the value of the determinant remains unchanged.

Examples: Use the properties of determinants to find the value of each determinant if it is known that

$$\begin{vmatrix} a & b & c \\ 5 & 3 & 1 \\ p & q & r \end{vmatrix} = 7.$$

a) $\begin{vmatrix} c & b & a \\ 1 & 3 & 5 \\ r & q & p \end{vmatrix}$

b) $\begin{vmatrix} p & q & r \\ 10 & 6 & 2 \\ a & b & c \end{vmatrix}$

c) $\begin{vmatrix} a-p & b-q & c-r \\ 5 & 3 & 1 \\ p & q & r \end{vmatrix}$

d) $\begin{vmatrix} p & q & r \\ a & b & c \\ -15 & -9 & -3 \end{vmatrix}$

e) $\begin{vmatrix} p+20 & q+12 & r+4 \\ -10 & -6 & -2 \\ a & b & c \end{vmatrix}$

f) $\begin{vmatrix} a & b & 0 \\ 5 & 3 & 0 \\ p & q & 0 \end{vmatrix}$