Sequences

A **sequence** $\{a_n\}$ is a list of numbers written in an explicit order. $\{a_n\}=\{a_1, a_2, a_3, \ldots, a_n \ldots\}$ a_1 is the first term, a_2 is the second term and so forth. A sequence never ends. The numbers in the list are called the **terms** of the sequence and a_n is the **nth term**. A sequence can be finite or infinite.

Example1: Defining a Sequence Explicitly

Find the first six terms and the 100th term of the sequence $\{a_n\}$ where $a_n = \frac{(-1)^n}{n^2+1}$

The sequence in example one was defined **explicitly**.

A second way of defining a sequence is **recursively** which assigns a value to the first (or the first few) term(s) and specifies the nth term by a formula or equation that involves one or more of the terms preceding it.

Example 2: Defining a Sequence Recursively

Definition Arithmetic Sequence

A sequence $\{a_n\}$ is an **arithmetic sequence** if it can be written in the form $\{a, a + d, a + 2d, \ldots,$ $\mathbf{a + (n - 1)d}$$, \ldots$ for some constant d. The number d is the **common difference**. Each term in an arithmetic sequence can be obtained recursively from its preceding term by adding d: $\mathbf{a_n = a_{n-1} + d}$ for all $n \geq 2$.

Example 3

Definition Geometric Sequence

A sequence $\{a_n\}$ is a **geometric sequence** if it can be written in the form $\{a, ar, ar^2, \ldots,$ $\mathbf{a \cdot r^{(n-1)}}$$, \ldots$ for some constant **r**. The number **r** is the **common ratio**. Each term in a geometric sequence can be obtained recursively from its preceding term by multiplying by r: $\mathbf{a_n = a_{n-1} \cdot r}$ for all $n \geq 2$.

Example 4

Example 5

Example 6:

Example 7:

Definition: Limit
Let L be a real number. The sequence $\{a_n\}$ has **limit L as n approaches** ∞ if, given any positive number ϵ, there is a positive number M such that for all n > M we have $|a_n - L| < \epsilon$
We write $\lim_{n\to\infty} a_n = L$ and say that the sequence **converges to L**. Sequences that do not have limits **diverge**.

Theorem 1: Properties of Limits
If L and M are real numbers and $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then
Sum Rule: $\lim_{n\to\infty}(a_n + b_n) = L + M$
Difference Rule: $\lim_{n\to\infty}(a_n - b_n) = L - M$
Product Rule: $\lim_{n\to\infty}(a_n b_n) = LM$
Constant Multiple Rule: $\lim_{n\to\infty}(c \cdot a_n) = c \cdot L$
Quotient Rule: $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{L}{M}$, $M \neq 0$

Example 8

Example 9

Theorem 2: The Sandwich Theorem for Sequences
If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ and if there is an integer N for which $a_n \leq b_n \leq c_n$ for all n > N, then $\lim_{n\to\infty} b_n = L$

Example 10 Using the Sandwich Theorem

Theorem 3 Absolute Value Theorem
Consider the sequence $\{a_n\}$. If $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$