Matrices – Solving Systems of Equations Using Row Operations

***Matrix*:** A rectangular array of numbers.
***Size of a Matrix*:** # of rows (m) $\times$ # of columns (n)

$$\begin{array}{c} \\ \text{row } 1 \\ \text{row } 2 \\ \vdots \\ \text{row } i \\ \vdots \\ \text{row } m \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{col } 1 & \text{col } 2 & \cdots & \text{col } j & \cdots & \text{col } n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Example: Name the size of $\begin{bmatrix} 4 & 3 \\ 2 & 6 \\ 1 & -5 \end{bmatrix}$

Each number a_{ij} of a matrix is called an ***entry*.** Each entry has two indices: i is the ***row index***, which tells what row the entry is on, and j is the ***column index***, which tells what column the entry is in. For example, a_{23} refers to the entry in the 2nd row and 3rd column.

***Augmented Matrix*:** A matrix used to represent a system of equations.

If we do not include the constants to the right of the equal sign, that is, to the right of the bar in the augmented matrix, the resulting matrix is called the ***coefficient matrix*** of the system.

Examples: Write the augmented matrix and coefficient matrix for each system of equations.

a) $\begin{cases} 3x - 4y = -6 \\ 2x - 3y = -5 \end{cases}$

b) $\begin{cases} 4x - 3y + 2z = 0 \\ 3x - z + 2 = 0 \\ x + y - 9 = 0 \end{cases}$

***Row Operations*:**

1. Switch the order of the rows.
2. Multiply or divide a row by a non-zero constant.
3. Add a multiple of one row to another row.

Examples: Apply the following row operations, in order, to the augmented matrix.

$\left[\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -5 & -4 \end{array}\right]$

a) $R_2 = -2r_1 + r_2$

b) $R_1 = 3r_2 + r_1$

Example: Find a row operation that will result in the matrix $\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array}\right]$ having a 0 in row 1, column 2.

***Row Echelon Form*:**

1. The first non-zero entry on each row is 1, and only zeros appear below it.
2. The first 1 in each row is to the right of the leading 1 in any row above.
3. Any rows that contain all 0's to the left of the vertical bar appear at the bottom.

***Reduced Row Echelon Form*:**
Entries are 0 above, as well as below, the leading 1's in each row.

***Consistent System*:** A system of equations with at least one solution.
***Inconsistent System*:** A system of equations with no solution.
***Dependent System*:** A system of equations with infinitely many solutions. (At least one of the variables *depends* on the values of the other variables.)

Examples: The reduced row echelon form of a system of equations is given. Write the system of equations corresponding to the given matrix. Determine whether the system is consistent or inconsistent. If it is consistent, give the solution.

a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

b) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$

c) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right]$

Solving Systems of Equations Using Row Operations:

- Perform row operations to get a leading 1 in row 1, column 1.
- Once you have a leading 1, perform row operations to get zeros above and below it.
- Work to get a leading 1 in row 2, column 2, then get zeros above and below it.
- Keep going, working top to bottom and left to right.
- Be careful and show your work! It is much better to do it right the first time than to go back and search for a mistake.

Examples: Solve each system of equations using matrix row operations. If the system has no solution, say that it is inconsistent.

a) $\begin{cases} \frac{1}{2}x + y = -2 \\ x - 2y = 8 \end{cases}$

b) $\begin{cases} 2x-3y-z=0 \\ -x+2y+z=5 \\ 3x-4y-z=1 \end{cases}$

c) $\begin{cases} w+x+y+z=4 \\ 2w-x+y=0 \\ 3w+2x+y-z=6 \\ w-2x-2y+2z=-1 \end{cases}$