The Hyperbola

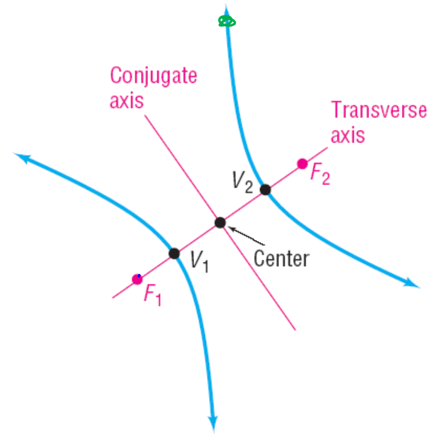
***Hyperbola*:** The collection of all points in the plane, the difference of whose distances from two fixed points, called the **foci,** is a constant.

***Transverse Axis*:** The line containing the foci.

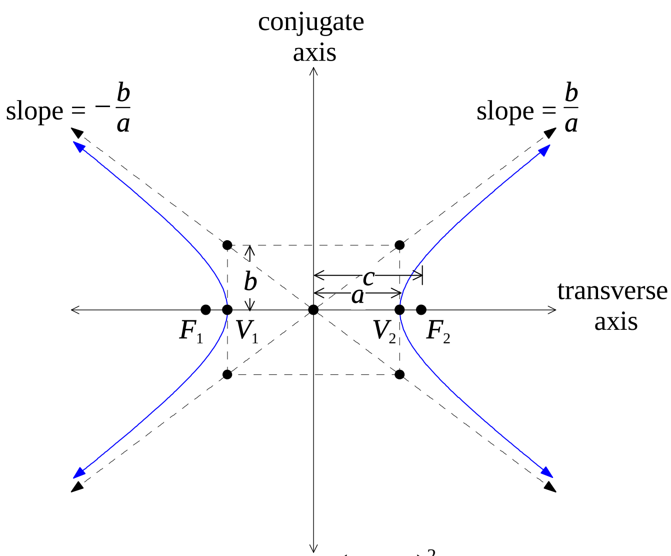
***Center*:** The midpoint of the line segment joining the foci.

***Conjugate Axis*:** The line through the center and perpendicular to the transverse axis.

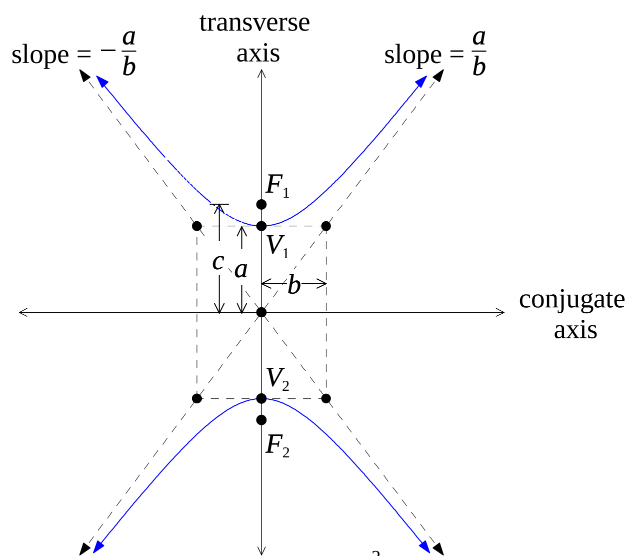
***Branches*:** The separate curves of the hyperbola. They are symmetric with respect to the transverse axis, conjugate axis, and center.



***Vertices*:** The points of intersection of the hyperbola and the transverse axis.



$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$
$$a^2+b^2=c^2$$



$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$
$$a^2+b^2=c^2$$

$a =$ distance from center to vertices

$c =$ distance from center to foci

b used to find the width of branches and slope of asymptotes

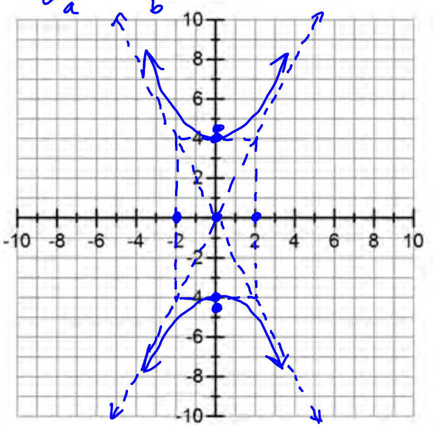
When finding the equations of the asymptotes, remember that $m=\dfrac{\text{change in } y}{\text{change in } x}$, or, in this case,

$m=\pm\dfrac{\sqrt{\#\text{ under } y^2 \text{ term}}}{\sqrt{\#\text{ under } x^2 \text{ term}}}$, then use point slope form $y-y_1=m(x-x_1)$ with the center (h,k) as (x_1,y_1).

Examples: Find the center, transverse axis, vertices, foci, and asymptotes. Graph each equation.

a) $\frac{y^2}{16} - \frac{x^2}{4} = 1$

$\updownarrow 4$ (a) $\leftrightarrow 2$ (b)



center: $(0,0)$

vertices: $(0,4)$ & $(0,-4)$

trans. axis: y-axis $(x=0)$

$c^2 = a^2 + b^2$

$c^2 = 16 + 4 = 20$

$c = \sqrt{20} = 2\sqrt{5} \approx 4.5$

foci: $(0, 2\sqrt{5})$ & $(0, -2\sqrt{5})$

asymptotes: $m = \frac{\updownarrow}{\leftrightarrow} = \pm\frac{4}{2} = \pm 2$

$y = \pm 2x$

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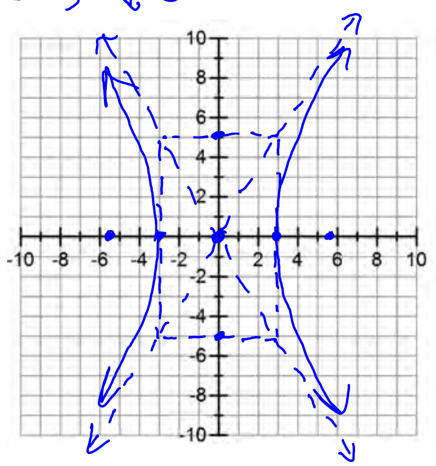
b) $\frac{x^2}{9} - \frac{y^2}{25} = 1$

$\leftrightarrow 3$ $\updownarrow 5$



center: $(0,0)$

vertices: $(3,0)$ & $(-3,0)$

trans. axis: x-axis $(y=0)$

$c^2 = a^2 + b^2 = 9 + 25 = 34$

$c = \sqrt{34} \approx 5.8$

foci: $(\sqrt{34}, 0)$ $(-\sqrt{34}, 0)$

asymptotes: $m = \frac{\updownarrow}{\leftrightarrow} = \pm\frac{5}{3}$

$y = \pm\frac{5}{3}x$

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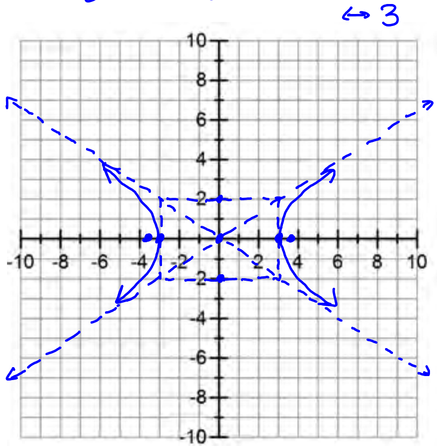
c) $\frac{4x^2 - 9y^2}{36} = \frac{36}{36}$

$\frac{x^2}{9} - \frac{y^2}{4} = 1$

$\leftrightarrow 3$ $\updownarrow 2$



center: $(0,0)$

vertices: $(3,0)$ & $(-3,0)$

transverse axis: x-axis $(y=0)$

$c^2 = a^2 + b^2 = 9 + 4 = 13$

$c = \sqrt{13} \approx 3.6$

foci: $(\sqrt{13}, 0)$ & $(-\sqrt{13}, 0)$

asymptotes: $m = \frac{\updownarrow}{\leftrightarrow} = \pm\frac{2}{3}$

$y = \pm\frac{2}{3}x$

point-slope form for line:

$y - y_1 = m(x - x_1)$

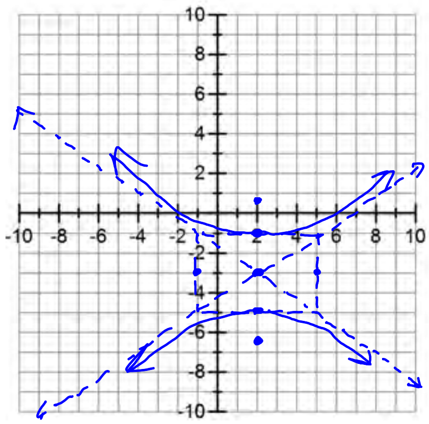
asymps: $y - k = m(x - h)$

asymptotes pass through (h, k)

x_1 y_1

d) $\dfrac{(y+3)^2}{4} - \dfrac{(x-2)^2}{9} = 1$

$\updownarrow 2$ $\leftrightarrow 3$



Center: $(2,-3)$ ← Careful! # w/ x first

Vertices: $(2,-1)$ & $(2,-5)$

trans. axis: $x = 2$

$c^2 = a^2 + b^2 = 4 + 9 = 13$

$c = \sqrt{13} \approx 3.6$

foci: $(2, -3+\sqrt{13})$ & $(2, -3-\sqrt{13})$

asymps: $m = \dfrac{\updownarrow}{\leftrightarrow} = \pm\dfrac{2}{3}$

$y + 3 = \pm\frac{2}{3}(x - 2)$

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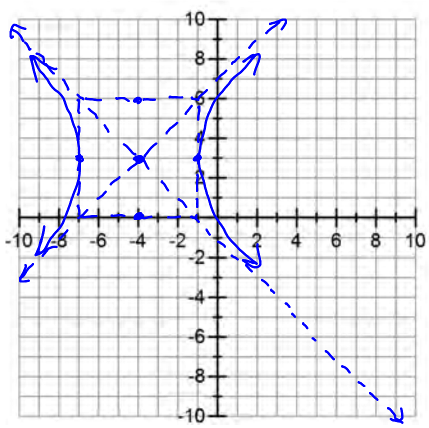
e) $\underbrace{(x+4)^2 - (y-3)^2}_{9} = \dfrac{9}{9}$



$\dfrac{(x+4)^2}{9} - \dfrac{(y-3)^2}{9} = 1$

$\leftrightarrow 3$ $\updownarrow 3$

center: $(-4, 3)$

vertices: $(-7,3)$ & $(-1,3)$

transverse axis: $y = 3$

$c^2 = a^2 + b^2 = 9 + 9 = 18$

$c = \sqrt{18} = 3\sqrt{2} \approx 4.2$

foci: $(-4-3\sqrt{2}, 3)$ & $(-4+3\sqrt{2}, 3)$

asymptotes: $m = \dfrac{\updownarrow}{\leftrightarrow} = \pm\dfrac{3}{3} = \pm 1$

$(y-3) = \pm(x+4)$

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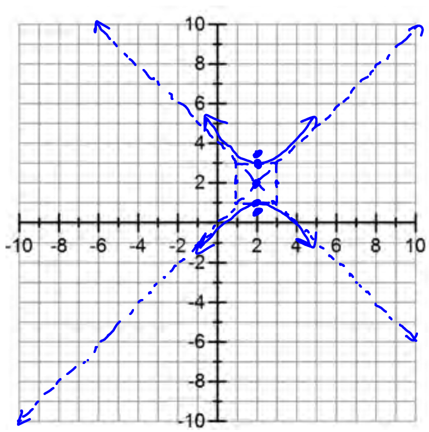
f) $y^2 - x^2 - 4y + 4x - 1 = 0$

$+(-x^2)$



$(y^2 - 4y) + (-x^2 + 4x) = 1$

Factor out negative!

$\frac{-4}{2} = -2$

$(-2)^2 = 4$

$(y^2 - 4y + 4) - (x^2 - 4x + 4) = 1 + 4 - 4$

Actually added -4

$(y-2)^2 - (x-2)^2 = 1$

$\updownarrow 1$ $\leftrightarrow 1$

Center: $(2,2)$

Vertices: $(2,3)$ & $(2,1)$

trans. axis: $x = 2$

$c^2 = a^2 + b^2 = 1 + 1 = 2$

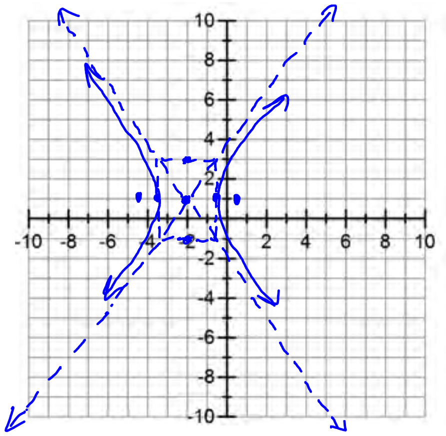
$c = \sqrt{2} \approx 1.4$

foci: $(2, 2+\sqrt{2})$ & $(2, 2-\sqrt{2})$

asymptotes: $m = \dfrac{\updownarrow}{\leftrightarrow} = \pm\dfrac{1}{1} = \pm 1$

$y - 2 = \pm(x - 2)$

g) $2x^2 - y^2 + 8x + 2y + 3 = 0$



Factor out 2   Factor out negative

$(2x^2+8x) + (-y^2+2y) = -3$

$\frac{4}{2}=2 \quad 2^2=4$

$-\frac{2}{2}=-1 \quad (-1)^2=1$

$2(x^2+4x+4) - (y^2-2y+1) = -3+8-1$

added $2(4)=8$ added -1

$\frac{2(x+2)^2-(y-1)^2}{4} = \frac{4}{4}$

$\frac{(x+2)^2}{2} - \frac{(y-1)^2}{4} = 1$

$\leftrightarrow \sqrt{2} \approx 1.4 \qquad \updownarrow 2$

center: $(-2,1)$

vertices: $(-2+\sqrt{2},1)$, $(-2-\sqrt{2},1)$

transverse axis: $y=1$

$c^2=a^2+b^2=2+4=6$

$c=\sqrt{6}\approx 2.4$

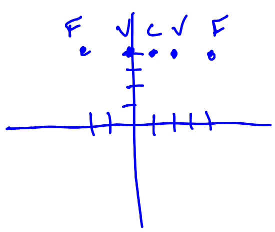
foci: $(-2+\sqrt{6},1)$, $(-2-\sqrt{6},1)$

asymptotes: $m=\frac{\updownarrow}{\leftrightarrow}=\pm\frac{2}{\sqrt{2}}=\pm\sqrt{2}$

$y-1=\pm\sqrt{2}(x+2)$

Examples: Write the equation of the hyperbola described.

a) Center at $(1,4)$; Focus at $(-2,4)$; Vertex at $(0,4)$



$\leftrightarrow$ $a=1$ (under x, $a^2=1$)

$c=3$

$a^2+b^2=c^2$

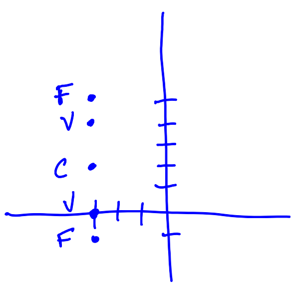
$1^2+b^2=3^2$

$b^2=3^2-1^2=8$ (under y, $b^2=8$)

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$\boxed{(x-1)^2 - \frac{(y-4)^2}{8} = 1}$

b) Focus at $(-3,5)$; Vertices at $(-3,4)$ and $(-3,0)$



center: $(-3,2)$

$\updownarrow$ $a=2$ (under y, $a^2=4$)

$c=3$

$2^2+b^2=3^2$

$b^2=3^2-2^2=5$

(under x, $b^2=5$)

$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

$\boxed{\frac{(y-2)^2}{4} - \frac{(x+3)^2}{5} = 1}$