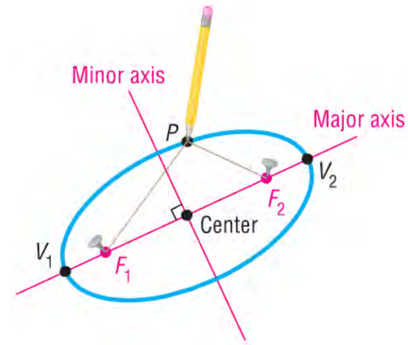
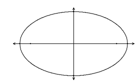
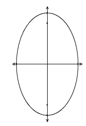
The Ellipse

Ellipse: The collection of all points in the plane, the sum of whose distances from two fixed points, called the ***foci***, F_1 and F_2, is a constant.

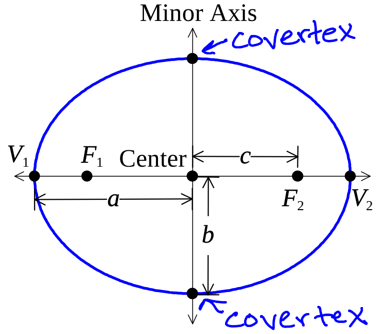


Major Axis: The line containing the foci.
Center: The midpoint of the line segment joining the two foci.
Minor Axis: The line through the center and perpendicular to the major axis.
Vertices: The points of intersection of the ellipse and the major axis.
Covertices: The points of intersection of the ellipse and the minor axis.

Standard Form of the Equation of an Ellipse with Center at (h,k)

Equation	Description	Picture
$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$ $a>b>0$ and $a^2-b^2=c^2$	Major axis parallel to *x*-axis	
$\dfrac{(x-h)^2}{b^2}+\dfrac{(y-k)^2}{a^2}=1$ $a>b>0$ and $a^2-b^2=c^2$	Major axis parallel to *y*-axis	

a **= Distance from center to vertices**
b **= Distance from center to covertices**
c **= Distance from center to foci**



vertices & foci are along major axis (the one that cuts ellipse in half the long way)

$\sqrt{\#}$ under x tells you how far ↔ from center
$\sqrt{\#}$ under y tells you how far ↕ from center
} a is the bigger #

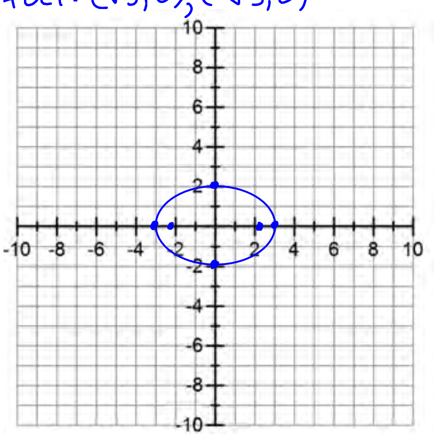
Examples: Find the center, foci, and vertices of each ellipse. Graph each equation.

a) $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$

$a^2 \to 9$, $4 \leftarrow b^2$
$\leftrightarrow 3$ (a) $\updownarrow 2$ (b)

$c^2=a^2-b^2$
$c^2=9-4=5$
$c=\sqrt{5}\approx 2.2$

center: $(0,0)$
vertices: $(3,0),(-3,0)$
foci: $(\sqrt{5},0),(-\sqrt{5},0)$

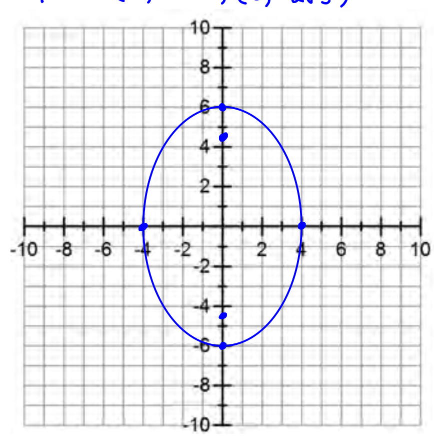


b) $\dfrac{x^2}{16}+\dfrac{y^2}{36}=1$

$b^2 \to 16$, $36 \leftarrow a^2$
$\leftrightarrow 4$ (b) $\updownarrow 6$ (a)

$c^2=a^2-b^2$
$c^2=36-16=20$
$c=\sqrt{20}=2\sqrt{5}\approx 4.5$

center: $(0,0)$
vertices: $(0,6),(0,-6)$
foci: $(0,2\sqrt{5}),(0,-2\sqrt{5})$



c) $\dfrac{(x+1)^2}{81}+\dfrac{(y-2)^2}{49}=1$

$a^2 \to 81$, $\leftrightarrow 9$ (a); $b^2 \to 49$, $\updownarrow 7$ (b)

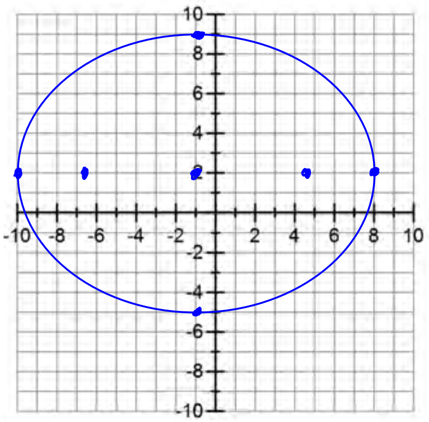
$c^2 = 81-49 = 32$

$c=\sqrt{32}=4\sqrt{2}\approx 5.7$

center: $(-1,2)$

vertices: $(-10,2)$ $(8,2)$

foci: $(-1+4\sqrt{2},2)$ $(-1-4\sqrt{2},2)$



d) $\dfrac{9(x-3)^2+(y+2)^2}{18}=\dfrac{18}{18}$

$\dfrac{(x-3)^2}{2}+\dfrac{(y+2)^2}{18}=1$

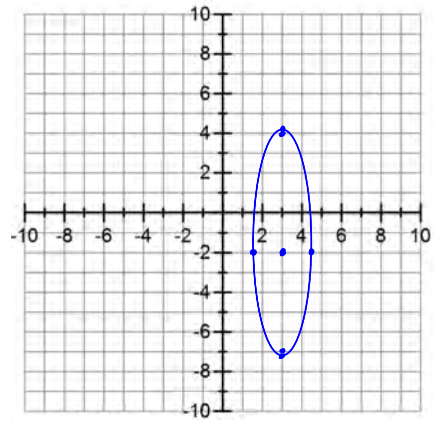
$b^2 \to 2$, $\leftrightarrow \sqrt{2}\approx 1.4$ (b); $a^2 \to 18$, $\updownarrow \sqrt{18}=3\sqrt{2}\approx 4.2$ (a)

$c^2=a^2-b^2$
$=18-2=16$
$c=\sqrt{16}=4$

center: $(3,-2)$

Vertices: $(3,-2+3\sqrt{2})$ $(3,-2-3\sqrt{2})$

foci: $(3,2)$ $(3,-6)$



e) $x^2+9y^2+6x-18y+9=0$

$(x^2+6x)+(9y^2-18y)=-9$

Factor out 9 before completing the square!

$(x^2+6x\qquad)+9(y^2-2y\qquad)=-9$

$\frac{6}{2}=3$, $3^2=9$

$(x^2+6x+9)+9(y^2-2y+1)=-9+9+9$

$\frac{-2}{2}=-1$, $(-1)^2=1$

actually added $9(1)=9$

$\dfrac{(x+3)^2+9(y-1)^2}{9}=\dfrac{9}{9}$

$\dfrac{(x+3)^2}{9}+(y-1)^2=1$

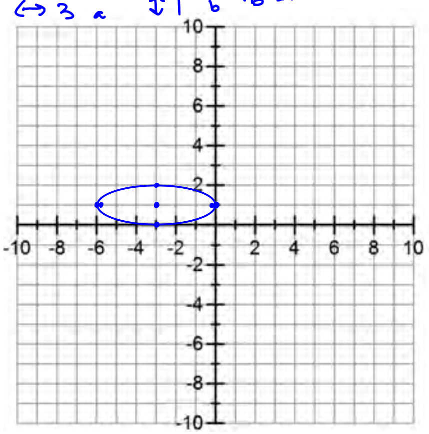
$a^2 \to 9$, $\leftrightarrow 3$ (a); $b^2=1$, $\updownarrow 1$ (b)

center: $(-3,1)$

vertices: $(-6,1)$, $(0,1)$

$c^2=a^2-b^2$
$c^2=9-1=8$
$c=\sqrt{8}=2\sqrt{2}\approx 2.8$

foci: $(-3+2\sqrt{2},1)$, $(-3-2\sqrt{2},1)$



f) $4x^2+y^2+4y=0$

$4x^2+(y^2+4y\qquad)=0$

$\frac{4}{2}=2$, $2^2=4$

$4x^2+(y^2+4y+4)=0+4$

$\dfrac{4x^2+(y+2)^2}{4}=\dfrac{4}{4}$

$x^2+\dfrac{(y+2)^2}{4}=1$

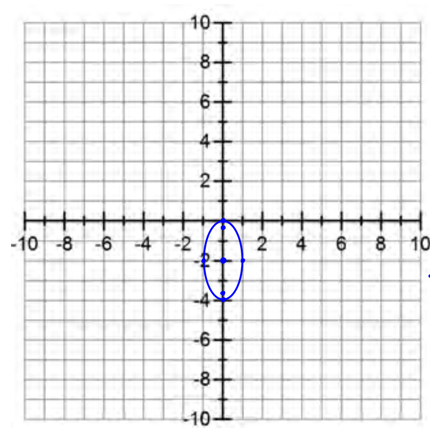
$b^2=1$, $\leftrightarrow 1$ (b); $a^2 \to 4$, $\updownarrow 2$ (a)

center: $(0,-2)$

Vertices: $(0,0)$, $(0,-4)$

$c^2=a^2-b^2$
$c^2=4-1=3$
$c=\sqrt{3}\approx 1.7$

foci: $(0,-2+\sqrt{3})$, $(0,-2-\sqrt{3})$



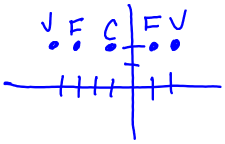
$a^2-b^2=c^2$
$a^2=b^2+c^2$
$b^2=a^2-c^2$

Examples: Write the equation of the ellipse having the given characteristics.

a) Foci at $(1,2)$ and $(-3,2)$; Vertex at $(-4,2)$

center @ midpoint

$\left(\frac{1+(-3)}{2},\ 2\right)=(-1,2)$



$a=3$ (center to vertex) ($\leftrightarrow$ a^2 under x)

$c=2$ (center to focus)

$b^2=a^2-c^2$
$=3^2-2^2=5$ (b^2 under y)

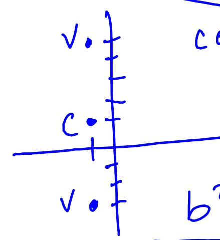
$$\boxed{\frac{(x+1)^2}{9}+\frac{(y-2)^2}{5}=1}$$

$a^2 \to 9$, $5 \leftarrow b^2$

b) Vertices at $(-1,5)$ and $(-1,-3)$; $c=1$

center @ midpoint

$\left(-1,\ \frac{5+(-3)}{2}\right)=(-1,1)$



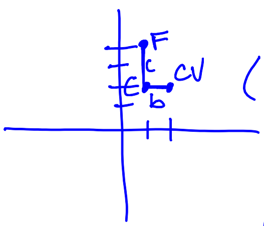
$a=4$ $\updownarrow$ (a^2 under y)

$b^2=a^2-c^2=4^2-1^2=15$ (b^2 under x)

$$\boxed{\frac{(x+1)^2}{15}+\frac{(y-1)^2}{16}=1}$$

$b^2 \uparrow$ $\uparrow a^2$

c) Center at $(1,2)$; Focus at $(1,4)$; Contains $(2,2)$



$b=1$ $\quad c=2$

($\leftrightarrow$ b^2 under x)

$a^2-b^2=c^2$

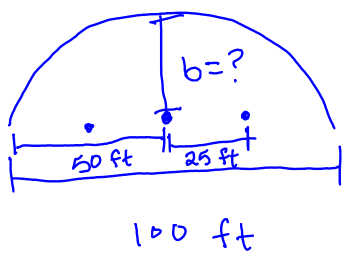
$a^2-1^2=2^2 \to a^2=2^2+1^2=5$

(a^2 under y)

$$\boxed{\frac{(x-1)^2}{1}+\frac{(y-2)^2}{5}=1}$$

$b^2=1 \rightarrow$, $5 \leftarrow a^2$

Example: A hall 100 feet in length is to be designed as a whispering gallery. If the foci are located 25 feet from the center, how high will the ceiling be at the center?



$a=50$ ft

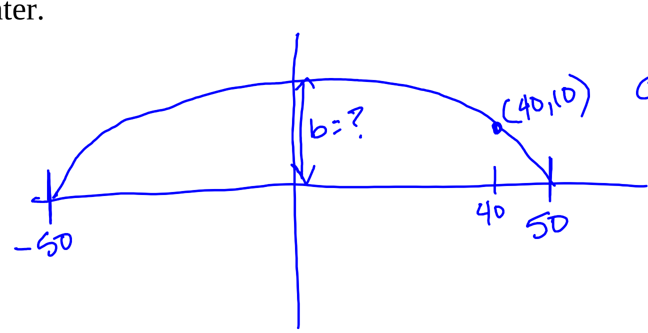
$c=25$ ft

$b^2=a^2-c^2$

$b^2=50^2-25^2=1875$

$b=\sqrt{1875}\approx \boxed{43.3 \text{ ft high}}$

Example: A bridge is to be built in the shape of a semielliptical arch and is to have a span of 100 feet. The height of the arch, at a distance of 40 feet from the center is to be 10 feet. Find the height of the arch at its center.



$a=50$

center: $(0,0)$

$$\frac{x^2}{50^2}+\frac{y^2}{b^2}=1$$

Goal: Find b

Plug in $(40,10)$: $\frac{40^2}{50^2}+\frac{10^2}{b^2}=1$

$\frac{1600}{2500}+\frac{100}{b^2}=1$

$-\frac{1600}{2500}$ $\quad -\frac{1600}{2500}$

$\frac{100}{b^2}=\frac{9}{25}$

$9b^2=2500$

$\sqrt{b^2}=\sqrt{\frac{2500}{9}}$

$b=\frac{50}{3}$ ft $\approx \boxed{16.7 \text{ ft}}$