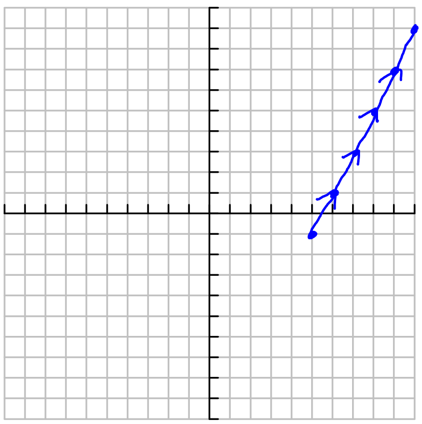
Parametric Equations

Sometimes, it is convenient to express both *x* and *y* as functions of a third variable, *t*. If $f(t)$ and $g(t)$ are both functions of *t*, where *t* is some interval of real numbers, then the equations $x = f(t)$ and $y = g(t)$ are called **parametric equations.** The variable t is called the **parameter.** If we think of *t* as time, then we know when each point of the graph is plotted.

Graphing Parametric Equations

1. Make a *t*, *x*, *y* table for the two equations.
2. Plot the ordered pairs of values of *x* and *y*.
3. Mark the **orientation** of the curve by using arrows to show the direction of the graph.

Example: Graph the parametric equations $x = t + 5$ and $y = 2t - 1$ for t in $[0,5]$.



t	x	y
0	(5,	-1)
1	(6,	1)
2	(7,	3)
3	(8,	5)
4	(9,	7)
5	(10,	9)

Eliminating the Parameter

1. Set one equation equal to t.
2. Substitute that equation in for t in the other equation.
3. Sometimes it is more convenient to use a trigonometric identity to eliminate the parameter.

Watch for problems like this:
$x = t + 6$ $\quad y = t^2 + 12t + 36$
$y = (t+6)^2$
$\boxed{y = x^2}$
Much easier than solving x equation for t & using substitution!

Examples: Eliminate the parameter and identify the graph of the parametric equation.

a) $x = 4t - 9,\ y = -t + 1,\ -\infty < t < \infty$

$+9 \quad +9$

$\frac{x+9}{4} = \frac{4t}{4}$

$t = \frac{1}{4}x + \frac{9}{4}$

$y = -\left(\frac{1}{4}x + \frac{9}{4}\right) + 1$

$y = -\frac{1}{4}x - \frac{9}{4} + 1$

$\boxed{y = -\frac{1}{4}x - \frac{5}{4}}$ line

b) $x = 2\sqrt{t},\ y = 8t + 6,\ 0 \le t < \infty$

$x \ge 0$ since $2\sqrt{t}$ will always be positive

$\frac{x}{2} = \sqrt{t}$

$t = \left(\frac{x}{2}\right)^2$

$t = \frac{x^2}{4}$

$y = 8\left(\frac{x^2}{4}\right) + 6;\ x \ge 0$

$\boxed{y = 2x^2 + 6;\ x \ge 0}$ half of a parabola

c) $x = 5\sin t,\ y = 5\cos t,\ -\infty < t < \infty$

1. Solve for sin t & cos t

$\sin t = \frac{x}{5} \qquad \cos t = \frac{y}{5}$

2. Square both sides

$\sin^2 t = \frac{x^2}{25} \qquad \cos^2 t = \frac{y^2}{25}$

3. Add equations

$\underbrace{\sin^2 t + \cos^2 t}_{1} = \frac{x^2}{25} + \frac{y^2}{25}$

4. Use $\sin^2 t + \cos^2 t = 1$

$\boxed{\frac{x^2}{25} + \frac{y^2}{25} = 1}$ (or $x^2 + y^2 = 25$) circle

d) $x = 2\sin\theta,\ y = 3\cos\theta,\ -\infty < \theta < \infty$

$\sin\theta = \frac{x}{2} \qquad \cos\theta = \frac{y}{3}$

$\sin^2\theta = \frac{x^2}{4} \qquad \cos^2\theta = \frac{y^2}{9}$

$\underbrace{\sin^2\theta + \cos^2\theta}_{1} = \frac{x^2}{4} + \frac{y^2}{9}$

$\boxed{\frac{x^2}{4} + \frac{y^2}{9} = 1}$ ellipse

Writing Parametric Equations for Line Segments

1. Write both parametric equations as linear functions: $x = m_1 t + b_1$, and $y = m_2 t + b_2$.
2. Substitute x and t values into the x equation to create a system of equations you can solve for m_1 and b_1.
3. Substitute y and t values into the y equation to create a system of equations you can solve for m_2 and b_2.

Examples:

Write parametric equations for the line segment starting at $(1,2)$ with $t=0$ and ending at $(8,10)$ with $t=1$.

$x = m_1 t + b_1$

$t=0,\ x=1:\ 1 = m_1(0) + b_1$

$b_1 = 1$

$t=1,\ x=8:\ 8 = m_1(1) + b_1$

$8 = m_1 + 1$

$m_1 = 7$

$x = 7t + 1$

$y = m_2 t + b_2$

$t=0,\ y=2:\ 2 = m_2(0) + b_2$

$b_2 = 2$

$t=1,\ y=10:\ 10 = m_2(1) + b_2$

$10 = m_2 + 2$

$m_2 = 8$

$y = 8t + 2$

$$\boxed{\begin{aligned} x &= 7t+1 \\ y &= 8t+2 \end{aligned}}$$

Write parametric equations for the line segment starting at $(-2,4)$ with $t=3$ and ending at $(5,-9)$ with $t=7$.

$x = m_1 t + b_1$

$t=3,\ x=-2:\ {}^{-1}(-2 = 3m_1 + b_1)$

$t=7,\ x=5:\ 5 = 7m_1 + b_1$

solve by elimination

$2 = -3m_1 - b_1$

$\frac{7}{4} = \frac{4m_1}{4}$

$m_1 = \frac{7}{4}$

$-2 = 3\left(\frac{7}{4}\right) + b_1$

$-2 = \frac{21}{4} + b_1$

$b_1 = -\frac{29}{4}$

$$\boxed{x = \frac{7}{4}t - \frac{29}{4}}$$

$y = m_2 t + b_2$

$t=3,\ y=4:\ {}^{-1}(4 = 3m_2 + b_2)$

$t=7,\ y=9:\ 9 = 7m_2 + b_2$

$-4 = -3m_2 - b_2$

$\frac{5}{4} = \frac{4m_2}{4}$

$m_2 = \frac{5}{4}$

$4 = 3\left(\frac{5}{4}\right) + b_2$

$4 = \frac{15}{4} + b_2$

$b_2 = \frac{1}{4}$

$$\boxed{y = \frac{5}{4}t + \frac{1}{4}}$$

Writing Parametric Equations for a Polar Equation

Use the equations $x = r\cos\theta$ and $y = r\sin\theta$. Replace r to obtain the parametric equations. When converting polar equations to parametric equations, θ acts as the parameter.

Example: Write parametric equations for the polar equation $r = 3\cos\theta$.

1. Start with $\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$ These are identities. They are always true.

2. Replace r with whatever r is equal to in the given polar equation (in this case, $3\cos\theta$)

$x = 3\cos\theta\cos\theta$

$y = 3\cos\theta\sin\theta$

3. Simplify

$$\boxed{\begin{aligned} x &= 3\cos^2\theta \\ y &= 3\cos\theta\sin\theta \end{aligned}}$$