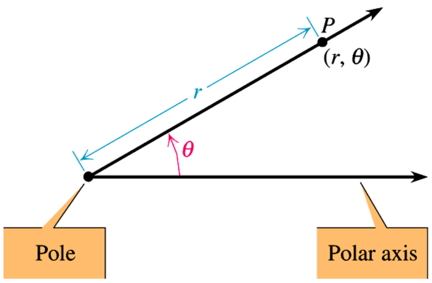
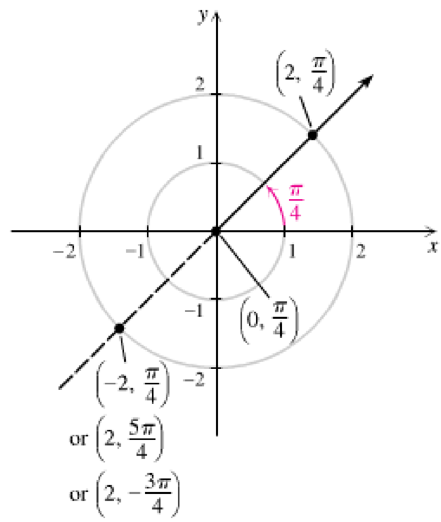
Polar Coordinates and Polar Equations



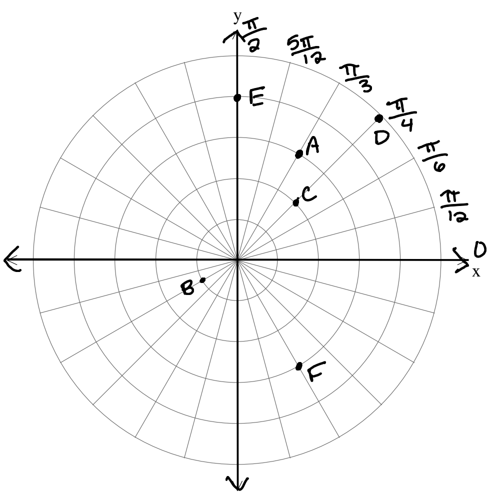
The polar coordinate system is based on a fixed point called the **pole** and a fixed axis called the **polar axis.** Points are represented by ordered pairs in the form (r,θ), where r is the **directed distance** from the pole and θ is an angle whose initial side is the polar axis and whose terminal side contains the point.
Typically, we choose the origin as the pole and the positive x-axis as the polar axis.



*To graph $(-r,\theta)$, you move in the opposite direction you would move to graph (r,θ).

Polar coordinates are not unique. The points $\left(-2,\frac{\pi}{4}\right)$, $\left(2,\frac{5\pi}{4}\right)$, and $\left(2,-\frac{3\pi}{4}\right)$ all name the same point.

Examples: Plot the points whose polar coordinates are given.
$A\left(3,\frac{\pi}{3}\right)$, $B\left(-1,\frac{\pi}{6}\right)$, $C\left(2,-\frac{7\pi}{4}\right)$, $D\left(-5,-\frac{3\pi}{4}\right)$, $E\left(4,\frac{\pi}{2}\right)$, $F\left(-3,\frac{2\pi}{3}\right)$



Polar-Rectangular Conversion Rules

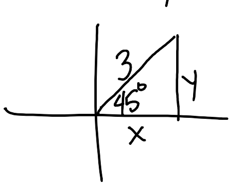
- To convert (r,θ) to rectangular coordinates (x,y), use $x = r\cos\theta$ and $y = r\sin\theta$.
- To convert (x,y) to polar coordinates (r,θ), use $r=\sqrt{x^2+y^2}$ and any angle θ in standard position whose terminal side contains (x,y).

Examples:

a) Convert $(3,45°)$ to rectangular coordinates.

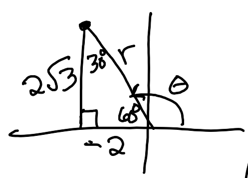
$x = 3\cos 45° = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$

$y = 3\sin 45° = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$



$\boxed{\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)}$

b) Convert $\left(-2,2\sqrt{3}\right)$ to polar coordinates.



$r = 4$

$\theta = 120°$

$\boxed{(4, 120°)}$

30°-60°-90° triangle
hyp = 2x short leg

Graphing Polar Equations

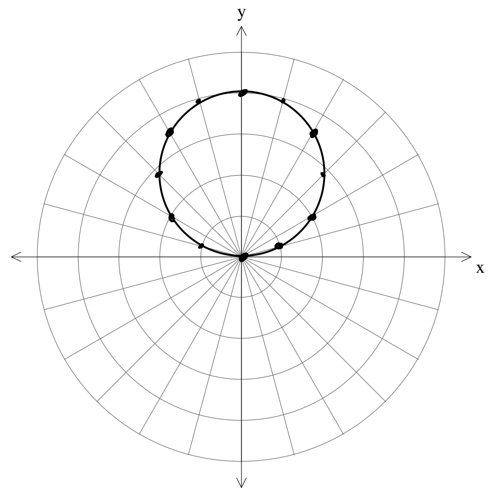
Examples: Sketch the graphs of the following:

a) $r = 4\sin\theta$

symmetric around y-axis

θ	r	
0°	0	$4\sin 0^\circ$
30°	2	$4\sin 30^\circ = 4\left(\frac{1}{2}\right) = 2$
60°	3.5	$4\sin 60^\circ = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$
90°	4	

circle

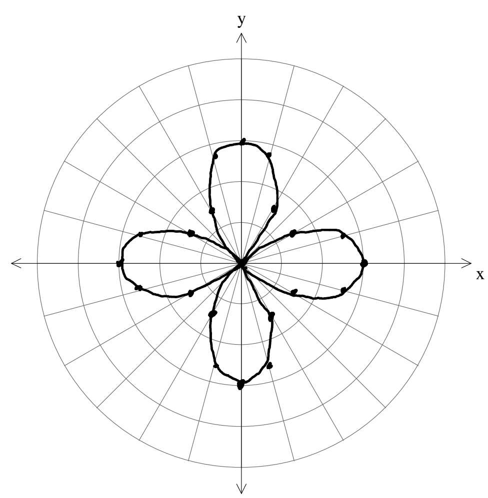


b) $r = 3\cos(2\theta)$

symmetric around x-axis

4-petal rose

r	θ
0°	3
30°	1.5
45°	0
60°	-1.5
90°	-3

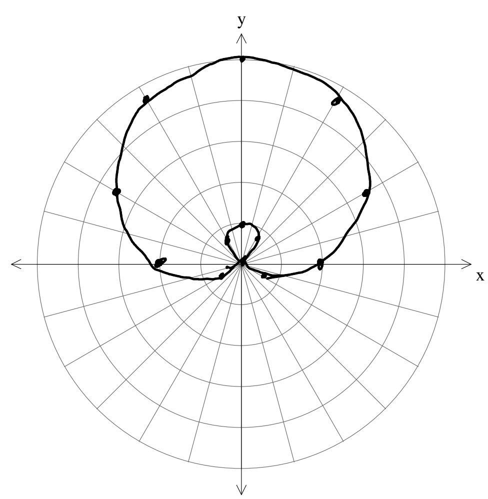


c) $r = 2 + 3\sin\theta$

symmetric around y-axis

limaçon with loop

θ	r
0°	2
30°	3.5
60°	4.6
90°	5
210°	0.5
240°	-0.6
270°	-1

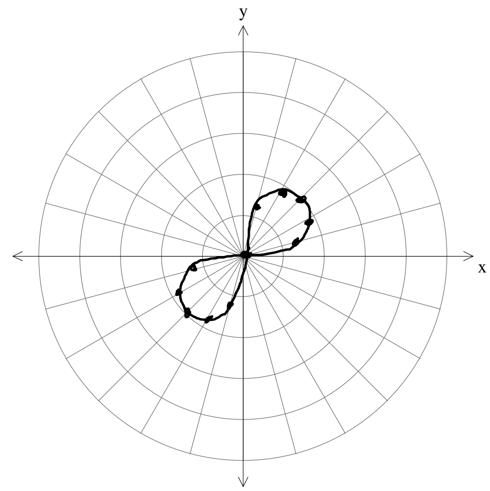


d) $\sqrt{r^2} = \pm\sqrt{4\sin(2\theta)}$ lemniscate

$r_1 = \sqrt{4\sin(2\theta)}$

$r_2 = -\sqrt{4\sin(2\theta)}$

θ	r
$0°$	0
$30°$	± 1.9
$45°$	± 2
$60°$	± 1.9
$90°$	0



e) $r = 5\sin(4\theta)$ 8-petal rose

θ	r
$0°$	0
$22.5°$	5
$67.5°$	-5
$112.5°$	5
$157.5°$	-5
$202.5°$	5
$247.5°$	-5
$292.5°$	5
$337.5°$	-5

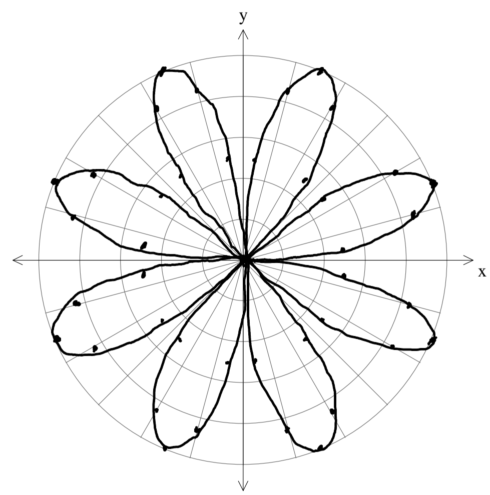
What angle has the highest sine?

$90°$

$4\theta = 90°$

$\theta = 22.5°$

peaks of petals @ 22.5° multiples



Examples: Convert equations from polar to rectangular form.

Look for r^2 ← change to x^2+y^2

$r\sin\theta$ ← change to y

$r\cos\theta$ ← change to x

a) Convert $r = 3\sin\theta$ to a rectangular equation.

Trick: multiply both sides by r

$r^2 = 3r\sin\theta$

$\boxed{x^2 + y^2 = 3y}$ circle

b) Convert $r = \dfrac{4}{1+\sin\theta}$ to a rectangular equation.

$r(1+\sin\theta) = 4$

$r + r\sin\theta = 4$

$r + y = 4$

$r = 4 - y$

$r^2 = (4-y)^2$

$x^2 + y^2 = 16 - 8y + y^2$

$-y^2 \qquad -y^2$

$\boxed{x^2 = 16 - 8y}$ parabola

c) Convert $r = 5\sec\theta$ to a rectangular equation.

$$r = \frac{5}{\cos\theta}$$

$$r\cos\theta = 5$$

$$\boxed{x = 5}$$ line

d) Convert $r = 5$ to a rectangular equation.

$$r^2 = 25$$

$$\boxed{x^2 + y^2 = 25}$$ circle

Examples: Convert equations from rectangular to polar form.

a) Convert $y = 7$ to a polar equation.

replace
x w/ $r\cos\theta$
y w/ $r\sin\theta$
x^2+y^2 w/ r^2

$$r\sin\theta = 7$$

$$r = \frac{7}{\sin\theta}$$

$$\boxed{r = 7\csc\theta}$$

b) Convert $y = -2x + 5$ to a polar equation.

$$r\sin\theta = -2r\cos\theta + 5$$

$$r\sin\theta + 2r\cos\theta = 5$$

$$r(\sin\theta + 2\cos\theta) = 5$$

$$\boxed{r = \frac{5}{\sin\theta + 2\cos\theta}}$$

c) Convert $x^2 + (y-1)^2 = 1$ to a polar equation.

$$x^2 + y^2 - 2y + 1 = 1$$
$$-1 \quad -1$$

$$r^2 - 2r\sin\theta = 0$$

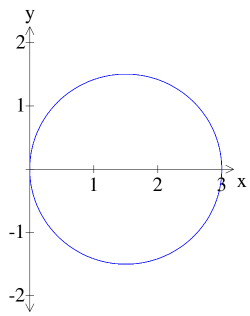
$$r(r - 2\sin\theta) = 0$$

$$r = 0 \qquad \boxed{r = 2\sin\theta}$$

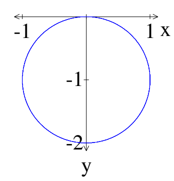
one of the solutions to $r=2\sin\theta$, so we can ignore it.

1) Lines through the origin are of the form $\theta = \alpha$.
Vertical lines are of the form $r = a\sec\theta$.
Horizontal lines are of the form $r = a\csc\theta$.

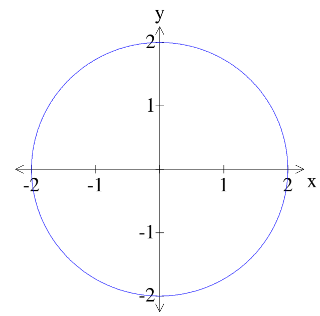
2) Circles come in three forms: $r = a\cos\theta$, $r = a\sin\theta$, and $r = a$.



$r = 3\cos\theta$

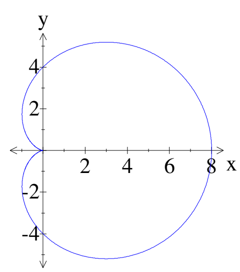


$r = -2\sin\theta$

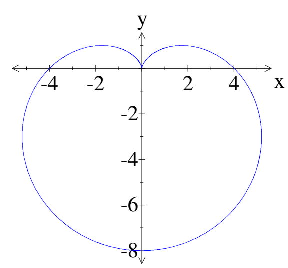


$r = 2$

3) Cardioids have the form $r = a \pm a\cos\theta$ or $r = a \pm a\sin\theta$.
Cardioids pass through the pole.

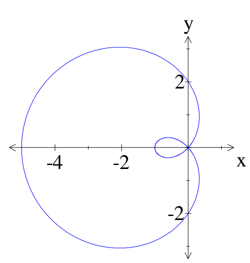


$r = 4 + 4\cos\theta$

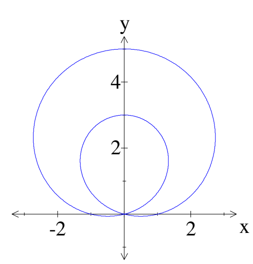


$r = 4 - 4\sin\theta$

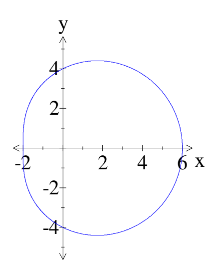
4) Limaçons have the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$.
Limaçons have an inner loop if $0 < a < b$ and have no inner loop if $0 < b < a$.
The graph of a limaçon with an inner loop passes through the pole twice.
The graph of a limaçon with no inner loop does not pass through the pole.



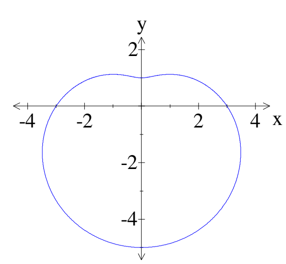
$r = 2 - 3\cos\theta$



$r = 1 + 4\sin\theta$



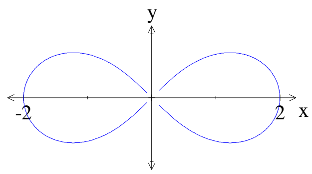
$r = 4 + 2\cos\theta$



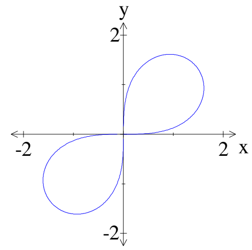
$r = 3 - 2\sin\theta$

5) Lemniscates have the form $r^2 = a^2\cos(2\theta)$ or $r^2 = a^2\sin(2\theta)$.

"Lemniscates are figure 8s"

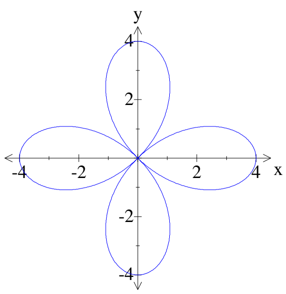


$r^2 = 4\cos(2\theta)$

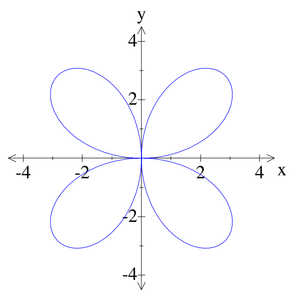


$r^2 = 4\sin(2\theta)$

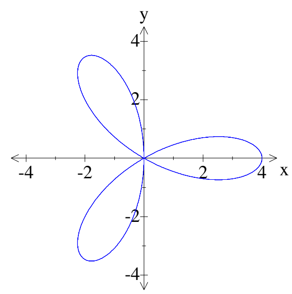
6) Roses have the form $r = a\cos(n\theta) + b$ and $r = a\sin(n\theta) + b$.
If n is even, there are $2n$ loops in the rose.
If n is odd, there are n loops in the rose.



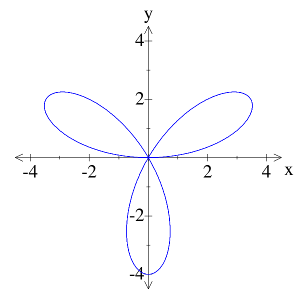
$r = 4\cos(2\theta)$



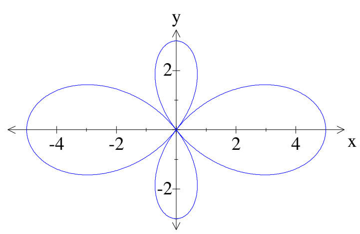
$r = 4\sin(2\theta)$



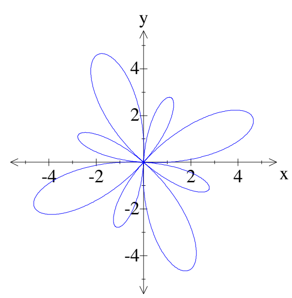
$r = 4\cos(3\theta)$



$r = 4\sin(3\theta)$



$r = 4\cos(2\theta) + 1$



$r = 4\sin(4\theta) + 1$