Powers and Roots of Complex Numbers

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is any positive integer, then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Examples:

a) Simplify $(1+i)^6$.

$r = \sqrt{2}$
$\theta = 45°$

$(\sqrt{2}\ cis\ 45°)^6$

$= \sqrt{2}^6\ cis\,(6\cdot 45°)$

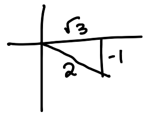
$\underbrace{\sqrt{2}\cdot\sqrt{2}}_{2}\cdot\underbrace{\sqrt{2}\cdot\sqrt{2}}_{2}\cdot\underbrace{\sqrt{2}\cdot\sqrt{2}}_{2}$

$(2^{\frac{1}{2}})^6 = 2^3$

$= 8(\cos 270° + i\sin 270°)$

$= 8(0 - i) = \boxed{-8i}$

b) Simplify $\left(\sqrt{3} - i\right)^4$.



$r = 2$
$\theta = -30°$

easier to multiply $-30°$ by 4 than $330°$ by 4

$(2\ cis\,(-30°))^4$

$= 2^4\ cis\,(-30°\cdot 4)$

$= 16\,(\cos(-120°) + i\sin(-120°))$

$= 16\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= \boxed{-8 - 8i\sqrt{3}}$

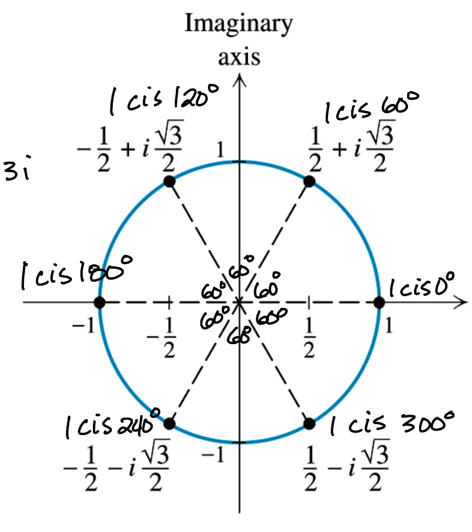
Roots of a Complex Number

How many square roots does 4 have? two $(2\ \&\ -2)$

How many square roots does -9 have? two → $x^2 = -9$, $x = \pm\sqrt{-9} = \pm 3i$

How many sixth roots does 1 have? six

It turns out that 1 has 6 sixth roots, and they are distributed evenly around the complex plane.



Sixth Roots of Unity

$r = 1 = 1^{1/6}$

$\frac{360°}{6} = 60°$ apart

The complex number $a + bi$ is an nth root of the complex number z if $(a+bi)^n = z$.

For any positive integer n, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots given by the expression $r^{1/n}(\cos\alpha + i\sin\alpha)$ where $\alpha = \dfrac{\theta + 360°k}{n}$ for $k = 0, 1, 2, \ldots, n-1$.

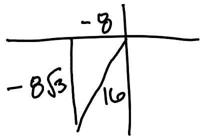
In radians, the roots are given by $r^{1/n}(\cos\alpha + i\sin\alpha)$ where $\alpha = \dfrac{\theta + 2k\pi}{n}$ for $k = 0, 1, 2, \ldots, n-1$.

The first of the n roots has an argument of $\dfrac{\theta}{n}$, and the other roots are spaced $\left(\dfrac{360}{n}\right)^{\circ}$ apart.

(The circle is divided evenly into n pieces.)

Examples:

a) Find all of the fourth roots of the complex number $-8-8i\sqrt{3}$.



$r=16$
$\theta=240^\circ$

$16^{1/4}=2$

$\dfrac{240^\circ+360^\circ k}{4}=60^\circ+90^\circ k$

$60^\circ, 150^\circ, 240^\circ, 330^\circ$

Trigonometric form: $2 \text{ cis } \alpha$, where $\alpha = 60^\circ, 150^\circ, 240^\circ, 330^\circ$

a+bi form:

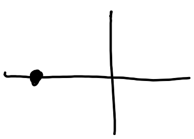
$2(\cos 60^\circ + i \sin 60^\circ)$
$2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$
$=\boxed{1+i\sqrt{3}}$

$2(\cos 150^\circ + i \sin 150^\circ)$
$2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$
$=\boxed{-\sqrt{3}+i}$

$2(\cos 240^\circ + i \sin 240^\circ)$
$2\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$
$=\boxed{-1-i\sqrt{3}}$

$2(\cos 330^\circ + i \sin 330^\circ)$
$2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$
$=\boxed{\sqrt{3}-i}$

b) Find all the cube roots of –125.



$r=125$
$\theta=180^\circ$

$125 \text{ cis } 180^\circ$

$125^{1/3}=5$

$\dfrac{180^\circ+360^\circ k}{3}=60^\circ+120^\circ k$

$60^\circ, 180^\circ, 300^\circ$

Trigonometric form: $5 \text{ cis } \alpha$, where $\alpha = 60^\circ, 180^\circ, 300^\circ$

a+bi form:

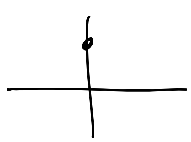
$5(\cos 60^\circ + i \sin 60^\circ)$
$=5\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$
$=\boxed{\frac{5}{2}+\frac{5\sqrt{3}}{2}i}$

$5(\cos 180^\circ + i \sin 180^\circ)$
$=5(-1+0i)$
$=\boxed{-5}$

$5(\cos 300^\circ + i \sin 300^\circ)$
$=5\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$
$=\boxed{\frac{5}{2}-\frac{5\sqrt{3}}{2}i}$

c) Find all complex solutions to $x^3-27i=0$.

$x^3=27i \rightarrow x=$ cube roots of $27i$



$r=27$
$\theta=90^\circ$

$27 \text{ cis } 90^\circ$

$27^{1/3}=3$

$\dfrac{90^\circ+360^\circ k}{3}=30^\circ+120^\circ k$

$30^\circ, 150^\circ, 270^\circ$

$3(\cos 30^\circ + i \sin 30^\circ)$
$3\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$
$\boxed{x=\frac{3\sqrt{3}}{2}+\frac{3}{2}i}$

$3(\cos 150^\circ + i \sin 150^\circ)$
$3\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$
$\boxed{x=-\frac{3\sqrt{3}}{2}+\frac{3}{2}i}$

$3(\cos 270^\circ + i \sin 270^\circ)$
$3(0-i)$
$\boxed{x=-3i}$