Powers and Roots of Complex Numbers

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is any positive integer, then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Examples:

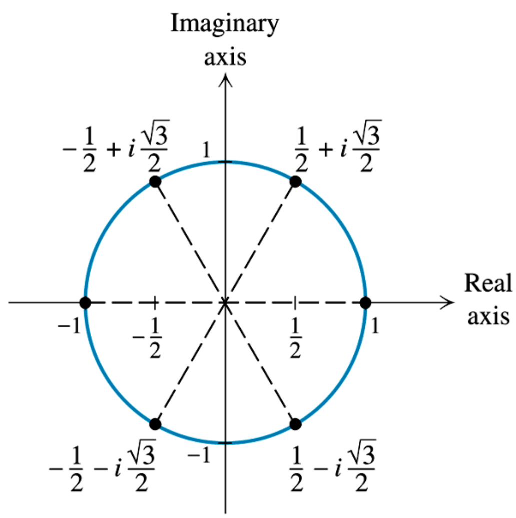
a) Simplify $(1+i)^6$.

b) Simplify $\left(\sqrt{3} - i\right)^4$.

Roots of a Complex Number

How many square roots does 4 have?
How many square roots does -9 have?
How many sixth roots does 1 have?

It turns out that 1 has 6 sixth roots, and they are distributed evenly around the complex plane.



Sixth Roots of Unity

The complex number $a+bi$ is an nth root of the complex number z if $(a+bi)^n = z$.

For any positive integer n, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots given by the expression $r^{1/n}(\cos\alpha + i\sin\alpha)$ where $\alpha = \dfrac{\theta + 360°k}{n}$ for $k = 0, 1, 2, ..., n-1$.

In radians, the roots are given by $r^{1/n}(\cos\alpha + i\sin\alpha)$ where $\alpha = \dfrac{\theta + 2k\pi}{n}$ for $k = 0, 1, 2, ..., n-1$.

The first of the n roots has an argument of $\dfrac{\theta}{n}$, and the other roots are spaced $\left(\dfrac{360}{n}\right)^{\circ}$ apart.

(The circle is divided evenly into n pieces.)

Examples:

a) Find all of the fourth roots of the complex number $-8-8i\sqrt{3}$.

b) Find all the cube roots of –125.

c) Find all complex solutions to $x^3 - 27i = 0$.