Antidifferentiation (Integration) by Parts

One of the most useful derivative formulas is the product rule, which is $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$

If we integrate with respect to "*x*" *we* get

$$\int\left(\frac{d}{dx}uv\right)dx = \int\left(u\frac{dv}{dx}\right)dx + \int\left(v\frac{du}{dx}\right)dx$$

Rearranging we get $\int u\frac{dv}{dx}dx = \int\frac{d}{dx}uv\,dx - \int v\frac{du}{dx}dx$

If we think of all the d's, dv's and dx's as differentials then we get (from the FTC: differentiation and integration are inverses of one another)

$$\int u\,dv = uv - \int v\,du$$

This is the integration by parts formula (Ibp). It expresses one integral in terms of another one. If we choose **u** and **dv** carefully, the second integral may be easier to integrate.

Notice the formula uses **u, v,** and **du**, but the original integral only has **u** and **dv**. So to use the formula we will need to differentiate **u** and antidifferentiate **dv**. This will help us carefully pick **u** and **dv**.

Let's try this on a problem.

Example: $\int x\cos x\,dx$ Let **u = x** and **dv = cos x dx**

This is an easy derivative (**du = dx**) easy antiderivative **v = sin x**

Formula:

$$\int u\,dv = uv - \int v\,du$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$\int x\cos x\,dx = x\sin x + \cos x + C$$

This isn't so bad, but what if we chose the wrong **u** and **dv**?

Exploration 1

LIPET: If you are wondering what to choose for ***u***, here is what we usually do. Our first choice is a natural logarithm (**L**), if there is one. If there isn't, we look for an inverse trig function (**I**). If there isn't one of these either, look for a polynomial (**P**). Still nothing, look for an exponential (**E**) or a trig function (**T**). That's the preference order: **L I P E T** In general, we want **u** to be something that simplifies when differentiated, and **dv** to be something that remains manageable when integrated.

$\int x^2 \cos x\,dx \qquad u = x^2 \qquad dv = \cos x\,dx$

$du = 2x\,dx \qquad v = \sin x$

$\int x^2 \cos x\,dx = \; x^2 \sin x - \int (\sin x) 2x\,dx$

$= x^2 \sin x - \int 2x \sin x\,dx$

$= x^2 \sin x - 2\int x \sin x\,dx \qquad u = x \; and \; dv = \sin x\,dx$

$du = dx \; and \; v = -\cos x$

$= x^2 \sin x - 2[x(-\cos x) - \int -\cos x\,dx]$

$= x^2 \sin x + 2x\cos x - 2\int \cos x\,dx$

$x^2 \sin x + 2x\cos x - 2\sin x + C$

On some problems, it seems it doesn't matter what you pick.

#17 $\int e^x \sin x\,dx$ Let u = sin x so du = cos x dx and let dv = e^x dx so v = e^x

$= e^x \sin x - \int e^x \cos x\,dx$ Let u = cos x so du = -sin x dx and let dv = e^x dx so v = e^x

$= e^x \sin x - [e^x \cos x - \int e^x \left(-\sin x\,dx\right)]$

$\int e^x \sin x\,dx = e^x \sin x - [e^x \cos x + \int e^x \sin x\,dx]$

$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$

*same *same

$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$

$$\int e^x \sin x\,dx = \frac{e^x \sin x - e^x \cos x}{2} + C$$

In this problem, because the derivatives and integrals of e^x and sin (x) keep reappearing, the original integral reappears and we can solve for it.

If we have integrals where the function we choose for u can be differentiated repeatedly until it becomes zero (i.e. power function) and dv can be easily integrated repeatedly, then we can do Tabular Integration. This is a fancy way to say integrating from a table. We will now use **u = f** and **dv = g**

On #5 we did $\int x^2 \cos x\,dx$ and used Integration by parts. Now let's use tabular.

f(x) and its derivatives	**g(x) and its integrals**
x^2 (+)	cos x
2x (-)	sin x
2 (+)	-cos x
0 (-)	-sin x

Once you have the table the integral is each version of f times each version of g that is one step lower with the attached sign + or - (multiply diagonally). Don't forget "Plus C"

$\int x^2 \sin x\,dx \; = \; x^2 \sin x + 2x\cos x - 2\sin x + C$

#23 $\int x^3 e^{-2x} dx$

#10 $\int t^2 \ln t\,dt$