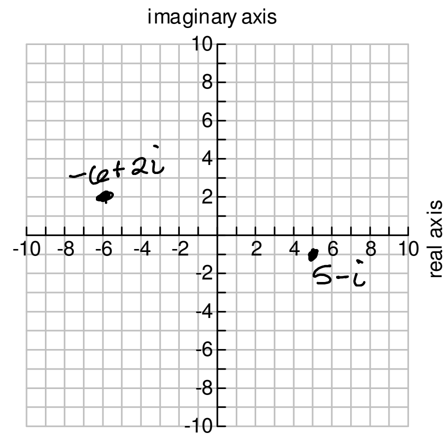
Trigonometric Form of Complex Numbers



The complex number $a+bi$ can be thought of as an ordered pair (a,b). We graph it on the **complex plane** where the horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis.**

Absolute Value or Modulus: $|a+bi| = \sqrt{a^2+b^2}$. $= r$ (The distance between the number and the origin on the complex plane.)

Examples: Graph each complex number and find its absolute value.

a) $5-i$

$|5-i| = \sqrt{5^2+1^2} = \sqrt{26}$

b) $-6+2i$

$|-6+2i| = \sqrt{6^2+2^2}$

$= \sqrt{40}$

$= 2\sqrt{10}$

Trigonometric Form of a Complex Number

If $z = a+bi$ is a complex number, then the trigonometric form of z is

$$z = r(\cos\theta + i\sin\theta), \text{ sometimes abbreviated } z = r\operatorname{cis}\theta,$$

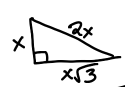
where r is called the **modulus** and θ is called the **argument,** defined as the angle in standard position whose terminal side contains the point (a,b).

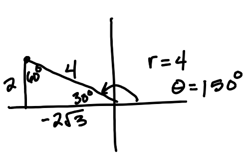
$$r = \sqrt{a^2+b^2}$$
$$a = r\cos\theta \text{ and } b = r\sin\theta.$$

We usually use the smallest possible nonnegative angle for θ.

Examples: Write each complex number in trigonometric form. Express θ in degrees.

a) $-2\sqrt{3}+2i$





$r=4$

$\theta=150°$

Hard way:

$r = \sqrt{(2\sqrt{3})^2+2^2}$

$= \sqrt{12+4} = 4$

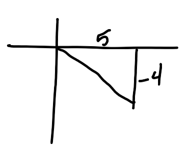
$\cos\theta = -\frac{2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$

$\sin\theta = \frac{2}{4} = \frac{1}{2}$

$\theta = 150°$

$\boxed{4(\cos 150° + i\sin 150°)}$

b) $5-4i$



$r = \sqrt{5^2+4^2}$

$= \sqrt{41}$

$\tan\theta = -\frac{4}{5}$

$\tan^{-1}\left(-\frac{4}{5}\right) \approx -38.7°$

$-38.7° + 360°$

$\approx 321.3°$

$\boxed{\sqrt{41}(\cos 321.3° + i\sin 321.3°)}$

Example: Write the complex number $12\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ in the form $a+bi$.

$$12\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$$

$$\boxed{-6\sqrt{2}+6i\sqrt{2}}$$

Product and Quotient of Complex Numbers

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]$$

Examples: Find the product and quotient using trigonometric form.

$$z_1 = 4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right),\quad z_2 = 8\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$

a) Find $z_1 z_2$

$4\cdot 8 \text{ cis}\left(\frac{\pi}{4}+\frac{\pi}{12}\right)$

$= 32 \text{ cis}\left(\frac{4\pi}{12}\right) = 32 \text{ cis}\left(\frac{\pi}{3}\right)$

$= 32\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)$

$= 32\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = \boxed{16+16i\sqrt{3}}$

b) Find $\frac{z_1}{z_2}$

$\frac{4}{8} \text{ cis}\left(\frac{\pi}{4}-\frac{\pi}{12}\right)$

$= \frac{1}{2} \text{ cis}\left(\frac{2\pi}{12}\right) = \frac{1}{2} \text{ cis}\left(\frac{\pi}{6}\right)$

$= \frac{1}{2}\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) = \boxed{\frac{\sqrt{3}}{4}+\frac{i}{4}}$

Complex Conjugates

The conjugate of $r(\cos(\theta)+i\sin(\theta))$ is $r(\cos(-\theta)+i\sin(-\theta))$

A complex number times its conjugate equals r^2.

Proof: $r(\cos\theta+i\sin\theta)\cdot r(\cos(-\theta)+i\sin(-\theta))$

$$= r^2(\cos(\theta-\theta)+i\sin(\theta-\theta))$$

$$= r^2(\cos 0 + i\sin 0)$$

$$= r^2(1+0i) = r^2$$

Example: Find the product of the following and its conjugate: $6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.

Conjugate: $6 \text{ cis}\left(-\frac{\pi}{3}\right)$

$$\left[6 \text{ cis}\left(\frac{\pi}{3}\right)\right]\cdot\left[6 \text{ cis}\left(-\frac{\pi}{3}\right)\right] = 6^2 = \boxed{36}$$